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A MODEL OF EDDY VISCOSITY AND EDDY DIFFUSIVITY OF HEAT

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Abstract— Eddy viscosity as a distribution of the probability density of the influence of a solid wall down into a fluid flow is derived. The expression for the wall influence is generally valid for the flow and heat transfer in smooth and rough channels and for surfaces in longitudinal flow. A model of the eddy viscosity of developed flow of a medium with constant properties in smooth tubes is presented. The coefficients of the model are found. Computed basic hydrodynamic characteristics are shown. An analogical model of the eddy diffusivity of heat is presented and its relation to the model of eddy viscosity is derived. Thermokinetic characteristics of the media (Pr = 0.72–10) for the uniform heat flux are computed. The model of eddy diffusivity is extended to liquid metals. Connecting the models together allows the influence of dissipated energy on the thermokinetic characteristics and the heat transfer coefficient for dissipated energy to be derived. The physical significance of the coefficients of the models are discussed and their relations to the mixing length and the quantities of vortex diffusion are indicated.

NOMENCLATURE

A, coefficient of the eddy viscosity model;
A_q, coefficient of the eddy diffusivity (of heat) model;
K, coefficient related to eddy viscosity;
K_q, coefficient related to eddy diffusivity of heat;
L, relative characteristic (mixing) length, l/r_w;
R, relative radius, r/r_w, or $1-Y$;
T, temperature;
U, relative velocity, u/u_s;
W, energy rate;
Y, relative coordinate, y/r_w, or $1-R$;
Fo, Fourier number, at/r_w^2;
Nu, Nusselt number, $2hr_w/\lambda$;
Pe, Peclet number, $Re\,Pr$;
Pr, Prandtl number, ν/a;
Pr_t, turbulent Prandtl number, $\varepsilon/\varepsilon_q$;
Re, Reynolds number, $2u_s r_w/\nu$;
Zh, Zhukowsky number, $\nu t/r_w^2$;
a, thermal diffusivity, $\lambda/\rho c_p$;
c_p, specific heat at constant pressure;
c_v, specific heat at constant volume;
c_Γ, velocity of circulation;
c_ω, peripheral velocity;
f, (Fanning) friction factor;
h, heat transfer coefficient;
l, mixing length;
ℓ, mean free path of molecules;
n, exponent;
p, pressure;
q, heat flux density;
$\tilde{q}$, energy generation per unit volume;
r, radius;
r^*, vortex radius;
r^{**}, recovery factor;
t, time;
$\bar{v}$, mean translation velocity of molecules;
x, y, coordinates;
y^+, dimensionless coordinate, u^*y/ν;
y^{++}, dimensionless coordinate, y^+Pr;
u, v, velocity in the x and y directions, respectively;
u^*, friction velocity, $(\tau_w/\rho)^{1/2}$;
u^+, dimensionless velocity, u/u^*.

Greek symbols

α, β, coefficients of the eddy viscosity model;
α_q, β_q, coefficients of the eddy diffusivity (of heat) model;
ε, eddy viscosity;
ε_q, eddy diffusivity of heat;
ν, kinematic viscosity;
τ, shear stress;
ρ, density of mass;
σ, standard deviation;
σ^2, dispersion;
ω, angle velocity;
δ, boundary layer thickness (relative);
μ, dynamic viscosity, $\nu\rho$;
Ω, vorticity;
Γ, circulation;
η, mean value of Rayleigh distribution;
λ, heat conductivity;
Θ, relative temperature, $(T-T_w)/(T_s-T_w)$;
ΔT^*, temperature difference for expression of the universal temperature profile, $q_w/\rho c_p u^*$;
ΔT^{**}, temperature difference necessary to remove the dissipated energy, q_w^{**}/h^{**};
φ, relative circulation velocity, c_Γ/u_s.

Other symbols

$\mathfrak{v}$, transverse mixing velocity;
$\mathcal{V}$, relative transverse mixing velocity, $\mathfrak{v}/u_s$;
$P(Y)$, probability;
$f(Y)$, distribution of probability density;
$F(Y)$, distribution function.

Subscripts and superscripts

a, molecular;

q, related to heat flux;
0, valid for the center line, initial;
s, mean, bulk;
t, turbulent;
$-t$, corresponding to time;
w, related to the wall;
y, related to the dimension coordinate;
K, modified;
T, related to the temperature;
Y, related to the relative coordinate;
ω, related to the angle velocity;
σ, corresponding to dispersion;
Γ, concerning circulation;
τ, related to shear stress;
ε, related to eddy viscosity;
ν, viscous;
in, inflexion;
max, maximum;
min, minimum;
$'$, fluctuating;
$\bar{}$, mean in time;
$^+$, dimensionless;
*, related to friction;
**, related to the dissipated energy;
***, related to the adiabatic wall temperature.

1. INTRODUCTION

THE LATEST developments in technology and science bring about an increased demand for accuracy in engineering computations. In the field of turbulent fluid flow and heat transfer there is a need to investigate thoroughly the local flow and thermokinetic conditions.

The characteristic quantities of turbulent fluid flow are most frequently expressed as a superposition of the mean value in time and the fluctuation component [1–4]. In engineering computations another quantity is introduced in accordance with this approach,

$$\varepsilon = -\frac{\overline{u'v'}}{\mathrm{d}u/\mathrm{d}y} = \frac{\tau_t}{\rho}\frac{1}{\mathrm{d}u/\mathrm{d}y}. \tag{1.1}$$

This quantity is analogous to molecular viscosity and is called the eddy viscosity or eddy diffusivity of momentum [2, 3]. In thermokinetics a similar transport quantity is used,

$$\varepsilon_q = -\frac{\overline{v'T'}}{\mathrm{d}T/\mathrm{d}y} = -\frac{q_t}{\rho c_p}\frac{1}{\mathrm{d}T/\mathrm{d}y}, \tag{1.2}$$

called the eddy diffusivity of heat.

In spite of the fact that the eddy viscosity and eddy diffusivity of heat occur in many models as the decisive quantities they have not been worked into a usable form satisfying both the boundary conditions and the experimental data of basic turbulent flow and thermokinetic characteristics till now [6]. The data necessary to determine the eddy viscosity which are being published are usually satisfactory only for a certain limited area of the fluid flow. The data given for the eddy diffusivity of heat are usually valid only for fluids for which the Prandtl numbers are kept within a very narrow interval.

The present state of basic knowledge of the distribution of the eddy viscosity along the radius of a circular pipe, or possibly along the normal to surface of a constant cross-section channel, may be summarized as follows:

(1) The course is a smooth function, its first derivation being continuous [6].

(2) At the center line of the channel, where $\mathrm{d}u/\mathrm{d}y = 0$, it has a finite value.

(3) On the channel wall it fades out while near the wall it changes with the cube of the distance from the wall [10].

(4) In a symmetric channel its course is also symmetric [12].

The course of the eddy diffusivity of heat shows identical properties. It differs only in that for high Prandtl number values ($Pr \gtrsim 7$) it increases near the wall with the fourth power of the distance [10].

The aim of the present paper is to express the eddy viscosity and the eddy diffusivity of heat in an analytic form suitable for application in solving problems of turbulent flow and heat transfer based on the statistical character of these quantities, in accordance with the basic experimental data. The model of the eddy viscosity and eddy diffusivity of heat presented here is limited to the stationary, fully developed flow of an incompressible fluid with constant thermophysical properties in a smooth pipe.

2. BASIC EQUATIONS OF TURBULENT FLOW IN A PIPE

The turbulent flow of the incompressible fluid is described by the momentum equation [14]

$$\frac{1}{r}\frac{\partial}{\partial r}\left[(\nu+\varepsilon)r\frac{\partial u}{\partial r}\right] = \frac{1}{\rho}\frac{\partial p}{\partial x}. \tag{2.1}$$

If the relative velocity $U = u/u_s$ is introduced at the bulk velocity defined by

$$u_s = \frac{2}{r_w^2}\int_0^{r_w} ur\,\mathrm{d}r \tag{2.2}$$

after the pressure gradient has been expressed using the Fanning friction factor as

$$-\frac{\partial p}{\partial x} = 2\frac{\tau_w}{r_w} = f\rho\frac{u_s^2}{r_w} \tag{2.3}$$

with the use of the relative coordinate $R = r/r_w = 1-Y$ it is possible to rewrite the momentum equation in a dimensionless form as

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}R}\left[\frac{\nu+\varepsilon}{\nu}R\frac{\mathrm{d}U}{\mathrm{d}R}\right] = -\frac{f}{2}Re, \tag{2.1a}$$

which must be completed by the normalization condition from the law of mass conservation,

$$\int_0^1 UR\,\mathrm{d}R = \tfrac{1}{2}. \tag{2.2a}$$

For the boundary conditions $R = 0 \Rightarrow \mathrm{d}U/\mathrm{d}R = 0$ and $R = 1 \Rightarrow U = 0$ the first formal integration of equation (2.1a) within the limits of $\langle 0, R\rangle$ follows the expression

$$\frac{\mathrm{d}U}{\mathrm{d}R} = -\frac{f}{2}Re\,\frac{\int_0^R R\,\mathrm{d}R}{R(\nu+\varepsilon)/\nu} = -\frac{f}{4}Re\frac{R}{(\nu+\varepsilon)/\nu}, \quad (2.4)$$

from which, after another formal integration within the limits $\langle R, 1\rangle$, another expression for velocity is obtained

$$U = \frac{f}{2}Re\int_R^1 \frac{\int_0^R R\,\mathrm{d}R}{R(\nu+\varepsilon)/\nu}\,\mathrm{d}R = \frac{f}{4}Re\int_R^1 \frac{R}{(\nu+\varepsilon)/\nu}\,\mathrm{d}R. \quad (2.5)$$

If equation (2.5) is substituted into equation (2.2a), after a formal integration, the following equation is obtained:

$$\frac{f}{2}Re\int_0^1 R\int_R^1 \frac{\int_0^R R\,\mathrm{d}R}{R(\nu+\varepsilon)/\nu}\,\mathrm{d}R\,\mathrm{d}R = \frac{1}{2}, \quad (2.6)$$

from which, when the sequence of the integration [3] is interchanged, after further adaptation follows

$$\frac{f}{4}Re = \frac{1}{\int_0^1 \frac{R^3}{(\nu+\varepsilon)/\nu}\,\mathrm{d}R}. \quad (2.6a)$$

It follows from equation (2.4), when $\varepsilon/\nu = 0$ is inserted for $R = 1$, that for the velocity gradient at the wall

$$\left(\frac{\mathrm{d}U}{\mathrm{d}R}\right)_{\mathrm{w}} = -\frac{f}{4}Re. \quad (2.7)$$

For the velocity at the general radius it is possible to write

$$U = \frac{\int_R^1 \frac{R}{(\nu+\varepsilon)/\nu}\,\mathrm{d}R}{\int_0^1 \frac{R^3}{(\nu+\varepsilon)/\nu}\,\mathrm{d}R}. \quad (2.5a)$$

The relations for the laminar flow follow directly from the above mentioned relations for $\varepsilon/\nu \equiv 0$.

By integrating equation (2.1) within the limits $\langle 0, r\rangle$, after multiplying it by the velocity gradient $\mathrm{d}u/\mathrm{d}r$, the following energy equation for the turbulent fluid flow is obtained [8]:

$$-\nu\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2 + \overline{u'v'}\frac{\mathrm{d}u}{\mathrm{d}r} = \left(\frac{r}{r_{\mathrm{w}}}\right)u^{*2}\frac{\mathrm{d}u}{\mathrm{d}r}. \quad (2.8)$$

This may be rewritten in a dimensionless form as

$$-\frac{\nu^2}{u^{*4}}\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2 + \nu\frac{\overline{u'v'}}{u^{*4}}\frac{\mathrm{d}u}{\mathrm{d}r} = \frac{\nu}{u^{*2}}\frac{r}{r_{\mathrm{w}}}\frac{\mathrm{d}u}{\mathrm{d}r}, \quad (2.8a)$$

$$W_\nu + W_t = W. \quad (2.8b)$$

The first term,

$$W_\nu = -\frac{\nu^2}{u^{*4}}\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2 = \left(\frac{R}{(\nu+\varepsilon)/\nu}\right)^2, \quad (2.8c)$$

expresses the direct viscous dissipated energy, while the second term

$$W_t = \nu\frac{\overline{u'v'}}{u^{*4}}\frac{\mathrm{d}u}{\mathrm{d}r} = \frac{\tau_t}{\tau_{\mathrm{w}}}\frac{R}{(\nu+\varepsilon)/\nu} = \frac{\varepsilon}{\nu}\left(\frac{R}{(\nu+\varepsilon)/\nu}\right)^2 = \frac{\varepsilon}{\nu}W_\nu \quad (2.8d)$$

expresses the turbulent energy production rate. The sum of these two terms gives the total energy rate,

$$W = \frac{\nu}{u^{*2}}\frac{r}{r_{\mathrm{w}}}\frac{\mathrm{d}u}{\mathrm{d}r} = \frac{R^2}{(\nu+\varepsilon)/\nu}. \quad (2.8e)$$

From equation (2.8d),

$$\frac{\varepsilon}{\nu} = \frac{W_t}{W_\nu}. \quad (2.9)$$

The relative eddy viscosity ε/ν then corresponds to the ratio of the local densities of turbulent energy production W_t and direct viscous dissipated energy W_ν. The mean relative eddy viscosity may be understood as the mean value of the density ratio of turbulent energy production and the direct viscous dissipated energy along the pipe radius

$$\left(\frac{\varepsilon}{\nu}\right)_{\mathrm{s}} = \int_0^1 \frac{\varepsilon}{\nu}\,\mathrm{d}R = \int_0^1 \frac{W_t}{W_\nu}\,\mathrm{d}R. \quad (2.10)$$

As in the differential equation (2.1a) for turbulent flow, which is the starting point here, the product $f\,Re$ appears. It is not possible to obtain from its solution an independent expression for f (or even for Re), but an expression in which both the quantities appear $(f/4)\,Re$, can be obtained.

With regard to equations (2.8e) and (2.6a), it is possible to explain the meaning of this complex using the relative energy density integral along the pipe cross-section

$$\frac{f}{4}Re = \frac{1}{\int_0^1 WR\,\mathrm{d}R} = -\frac{1}{2}\frac{\partial p}{\partial x}\frac{r_{\mathrm{w}}^2}{\mu u_{\mathrm{s}}} = \frac{\tau_{\mathrm{w}}r_{\mathrm{w}}}{u_{\mathrm{s}}\mu} = \frac{\tau_{\mathrm{w}}}{\rho}\frac{r_{\mathrm{w}}}{u_{\mathrm{s}}\nu} \quad (2.6a)$$

and the expression for the velocity, according to the equation (2.8c), using the relative direct viscous dissipated energy density

$$U = \frac{f}{4}Re\int_R^1 W_\nu^{1/2}\,\mathrm{d}R = \frac{f}{4}Re\int_R^1 \frac{W}{R}\,\mathrm{d}R. \quad (2.5b)$$

3. EDDY VISCOSITY MODEL

Let us take a random variable, $\mathscr{S}$, which acquires the value of the relative reach of the solid wall influence into the fluid flow. This variable has, in a general sense, a certain distribution function

$$F(Y) = P(\mathscr{S} \leqslant Y) \tag{3.1}$$

giving the probability that the influence will reach as far as the distance Y. With this distribution function, a certain probability density function $f(Y)$ is associated by [33, 34]

$$\mathrm{d}F(Y) = f(Y)\,\mathrm{d}Y = P(Y < \mathscr{S} \leqslant Y+\mathrm{d}Y), \tag{3.2}$$

stating the probability that the reach distance will acquire the value Y in the elementary interval $\langle Y, Y+\mathrm{d}Y\rangle$. The probability that the influence will extend past the distance Y, or that the influence will continue to the distance Y, is expressed by

$$P(\mathscr{S} > Y) = 1 - F(Y). \tag{3.3}$$

The conditional probability $\varphi(Y)$ that the influence will be damped in an elementary section $\mathrm{d}Y$, if it reaches as far as the distance Y, may be expressed by the ratio of the probabilities $P(Y < \mathscr{S} \leqslant Y+\mathrm{d}Y)$ and $P(\mathscr{S} > Y)$ [33, 34]

$$\varphi(Y) = P(Y < \mathscr{S} \leqslant Y+\mathrm{d}Y/\mathscr{S} > Y) = \frac{P(Y < \mathscr{S} \leqslant Y+\mathrm{d}Y)}{P(\mathscr{S} > Y)}. \tag{3.4}$$

By inserting equations (3.2) and (3.3) into equation (3.4), and after further adaptation, a differential equation can be obtained

$$F'(Y) = \varphi(Y)[1 - F(Y)]. \tag{3.5}$$

By solving this equation within the interval $\langle 0, Y\rangle$, with the boundary condition $Y = 0 \Rightarrow F(Y) = 0$, a general form for the distribution function is obtained

$$F(Y) = 1 - \exp\left[-\int_0^Y \varphi(Y)\,\mathrm{d}Y\right]. \tag{3.6}$$

Let us base our further considerations on the fact that the probability of the influence of the wall vanishing in the elementary section $\mathrm{d}Y$, when the distance Y has been reached, is directly proportional to this distance Y. By placing

$$\varphi(Y) = \frac{2}{\alpha}Y, \quad \alpha > 0; \quad Y \geqslant 0, \tag{3.7}$$

the exponential term in equation (3.6) will have the form

$$\int_0^Y \varphi(Y)\,\mathrm{d}Y = \int_0^Y \frac{2}{\alpha}Y\,\mathrm{d}Y = \frac{Y^2}{\alpha}, \tag{3.8}$$

from which an expression for the distribution of probability density of the wall influence follows

$$f(Y) = \frac{2}{\alpha}Y\mathrm{e}^{-Y^2/\alpha}, \tag{3.9}$$

which is, in mathematical statistics, called the Rayleigh distribution [34]. The corresponding distribution function has the form

$$F(Y) = 1 - \mathrm{e}^{-Y^2/\alpha}. \tag{3.10}$$

The mean value of the Rayleigh distribution (the first general moment) is given by [11, 34]

$$\eta = \tfrac{1}{2}(\pi\alpha)^{1/2}. \tag{3.11}$$

The dispersion (the second central moment) is given by [11, 34]

$$\sigma^2 = \frac{4-\pi}{4}\alpha. \tag{3.12}$$

For a more general expression of the function $\varphi(Y)$

$$\varphi(Y) = \frac{\beta}{\alpha}Y^{\beta-1} \quad Y \geqslant 0, \quad \alpha > 0, \quad \beta > 0, \tag{3.13}$$

the so-called Weibull distribution can be obtained for the probability density [33]

$$f(Y) = \frac{\beta}{\alpha}Y^{\beta-1}\mathrm{e}^{-Y^\beta/\alpha} \tag{3.14}$$

with the distribution function

$$F(Y) = 1 - \mathrm{e}^{-Y^\beta/\alpha} \tag{3.15}$$

where α is a parameter of the wall influence (the width of the distribution) and β is a parameter of the distribution form (for a Rayleigh distribution $\beta = 2$). For the Weibull distribution there is a corresponding mean value expressed by the Γ function [11, 34]

$$\eta = \alpha^{1/\beta}\Gamma\left(1 + \frac{1}{\beta}\right) \tag{3.16}$$

and a corresponding dispersion, expressed also by the Γ function [11, 34]

$$\sigma^2 = \alpha^{2/\beta}\left[\Gamma\left(1+\frac{2}{\beta}\right) - \Gamma^2\left(1+\frac{2}{\beta}\right)\right]. \tag{3.17}$$

The idea of the statistical distribution of the reach of the solid wall influence into the fluid flow, expressed by a Rayleigh or Weibull distribution, may be extended to rough surfaces. In this case it is necessary to use, instead of the two-parameter distribution, a distribution defined by three parameters, where the third parameter Y_w signifies the minimum reach of the wall influence for all cases (the distribution parameter in relation to the origin). This reflects the shift of the flow from the wall and is directly related to the value of the relative wall roughness

$$f(Y) = \frac{2}{\alpha}(Y - Y_w)\mathrm{e}^{-(Y-Y_w)^2/\alpha}, \tag{3.18a}$$

$$f(Y) = \frac{\beta}{\alpha}(Y - Y_w)^{\beta-1}\mathrm{e}^{-(Y-Y_w)^\beta/\alpha}. \tag{3.18b}$$

As equations (3.9) and (3.10) or (3.14) and (3.15) and (3.18a,b), which express the probability of the wall influence reach into the fluid flow, were derived in a quite general manner, their validity is also general. The distance from the wall where $P \to 1$ is the limit of the turbulent boundary layer (in a general sense of both the hydrodynamic and thermal) or the boundary layer relevant to the mass diffusion. The individual coefficients (α, β, Y_w) which occur in the above-mentioned equations are dependent on the particular geometric and hydrodynamic or thermal boundary conditions or on the conditions of mass diffusion.

If the relative distance $Y = y/r_w$ from the wall is replaced by the absolute distance y (m), or if it is related to a characteristic length other than the cross-dimension of the channel, the expressions are also valid for the case when the medium flows along the surfaces. In a channel, the value $P \to 1$ corresponds to the reach of one wall influence towards the opposite wall. In case of not fully developed convection, or in the entrance region, the probability acquires the value $P \to 1$ at a shorter distance from the wall ($Y < 2$).

The character of the eddy viscosity in a smooth pipe, specified in the Introduction under the items (1) and (3), satisfies the linear combination of the probability densities of the range of wall influence for the relevant solutions of the differential equation (3.5), namely the difference of the two partial stochastic processes with the Rayleigh distribution of probability density.

For the relative eddy viscosity the following equation may be written:

$$\frac{\varepsilon}{\nu} \sim Af(Y) - A_w f_w(Y), \tag{3.19}$$

where

$$f(Y) = \frac{2}{\alpha} Y e^{-Y^2/\alpha}; \quad f_w(Y) = \frac{2}{\alpha_w} Y e^{-Y^2/\alpha_w}, \quad \alpha > \alpha_w.$$

With regard to the symmetry of the problem [items (2) and (4)] it is necessary in the mathematical description of the model of eddy viscosity in a pipe to take into account the opposite symmetric branch with the coordinate $(2-Y)$. As at the point $Y = 2$ (on the opposite wall) the Rayleigh distribution is not exactly zero, corrective terms may be introduced which represent negative mirror-like (symmetric) functions outside the interval $Y \in \langle 0, 2 \rangle$, and the terms smaller than $(2/\alpha)4\,e^{-4^2/\alpha}$ or $(2/\alpha_w)4\,e^{-4^2/\alpha_w}$ may be neglected.

For the relative eddy viscosity the following expression may then be written:

$$\begin{aligned}\frac{\varepsilon}{\nu} = {} & \frac{2A}{\alpha}[Y e^{-Y^2/\alpha} + (2-Y)\, e^{-(2-Y)^2/\alpha} - (2+Y) \\ & \times e^{-(2+Y)^2/\alpha} - (4-Y)\, e^{-(4-Y)^2/\alpha}] \\ & - \frac{2A_w}{\alpha_w}[Y e^{-(Y^2/\alpha_w)} + (2-Y)\, e^{-(2-Y)^2/\alpha_w} - (2+Y) \\ & \times e^{-(2+Y)^2/\alpha_w} - (4-Y)\, e^{-(4-Y)^2/\alpha_w}] \\ = {} & \frac{2A}{\alpha} G(Y) - \frac{2A_w}{\alpha_w} G_w(Y).\end{aligned} \tag{3.20}$$

The analytic function expressed by the difference of two Rayleigh distributions is, at a very small distance from the wall, proportional to the third power of the distance from the wall

$$\frac{\varepsilon}{\nu} = KY^3. \tag{3.21}$$

From the development of the basic terms of equation (3.20) (which contain only Y) into power series, a relation between the coefficients A, α, A_w, α_w in that expression and the coefficient K in the equation (3.21) can be obtained

$$K = \frac{2(A-A_w)}{\alpha\alpha_w}. \tag{3.22}$$

From the condition of zero derivative on the wall ($Y = 0$) which results from the necessity of coincidence between the functions according to the equations (3.21) and (3.20), the following relationship between the coefficients in equation (3.20) will be obtained:

$$\frac{A}{\alpha} = \frac{A_w}{\alpha_w}. \tag{3.23}$$

This allows the expression for the relative eddy viscosity to be simplified to

$$\begin{aligned}\frac{\varepsilon}{\nu} = {} & \frac{2A}{\alpha}[Y(e^{-Y^2/\alpha} - e^{-Y^2/\alpha_w}) \\ & + (2-Y)(e^{-(2-Y)^2/\alpha} - e^{-(2-Y)^2/\alpha_w}) \\ & - (2+Y)(e^{-(2+Y)^2/\alpha} - e^{-(2+Y)^2/\alpha_w}) \\ & - (4-Y)(e^{-(4-Y)^2/\alpha} - e^{-(4-Y)^2/\alpha_w})] \\ = {} & \frac{2A}{\alpha} H(Y).\end{aligned} \tag{3.20a}$$

Equation (3.20a) satisfies all the characteristic features of eddy viscosity specified in the Introduction to this paper. The function according to equation (3.20a) has its maximum point, as well as two inflexion points, in the interval $Y \in \langle 0, 1 \rangle$. The minimum point is situated at the center line $Y = 1$ and on the wall.

As the Rayleigh distribution is a normalized function for which the following relation is valid:

$$\int_0^\infty f(Y)\,dY = \int_0^\infty \frac{2}{\alpha} Y e^{-Y^2/\alpha}\,dY = 1, \tag{3.24}$$

and as at the point $Y = 2$ (on the wall) it differs only slightly from zero, it may be considered, with no loss of accuracy, that

$$\int_0^2 f(Y)\,dY \doteq 1, \tag{3.24a}$$

and with regard to the symmetry of the functions in equation (3.20) that

$$\int_0^1 f(Y)\,dY + \int_0^1 f(2-Y)\,dY = \int_0^2 f(Y)\,dY = 1. \tag{3.24b}$$

An analogous consideration is also possible for the function $f_w(Y)$. The coefficients A and A_w then correspond to mean functional values and their difference

$$A - A_w = A_t = \left(\frac{\varepsilon}{\nu}\right)_s \tag{3.25}$$

corresponds directly to the mean value of the relative eddy viscosity along the pipe radius. For the coefficient

K in equation (3.22) it is possible to write, with regard to the equations (3.23) and (3.25),

$$K = \frac{2A_t}{\alpha\alpha_w} = \frac{2A}{\alpha\alpha_w}\left(1 - \frac{\alpha_w}{\alpha}\right) = \frac{2A}{\alpha}\left(\frac{1}{\alpha_w} - \frac{1}{\alpha}\right). \quad (3.22a)$$

In equation (3.20) there are four coefficients, namely A, α, α_w and A_w; for the application of the model it is necessary to know their values.

Equation (3.23) enables the number of unknown coefficients to be limited to three in equation (3.20a). Thus we need to know either three experimental data points or three physical conditions. For the individual flow regimes, defined by the Reynolds number Re, we may consider the available literature values of the Fanning friction factor f and the velocity U_0 at the center line, which may also be simply determined experimentally. As the third necessary datum the value of the arbitrary radial velocity may be used. In the first place we may take into account the radius on which the local velocity is equal to the bulk velocity, i.e. $U = 1$ (given in ref. [9]).

As the third condition, besides the experimental data for f or $(f/4)Re$ and the velocity U_0, the energy bond between the stochastic processes expressed by the probability distribution $f(Y)$ and $f_w(Y)$ (i.e. the principle of maximum energy degradation) may be used to find the values of the coefficients in equation (3.20a). The condition of maximum energy degradation is satisfied for the flow regime in question for the minimum mean value of the ratio of production of turbulent energy density to direct viscous dissipated energy along the pipe radius which equals, according to equation (2.10), the mean value of the relative eddy viscosity and, according to equation (3.25), also equals the value of the coefficient A_t

$$A_t = \left(\frac{\varepsilon}{\nu}\right)_s = \int_0^1 \frac{W_t}{W_v}\,dR = \int_0^1 \frac{\varepsilon}{\nu}\,dR \sim \min. \quad (3.26)$$

The system of equations (2.5) and (2.6a) for given couples of values of U_0 and $(f/4)\,Re$, corresponding to the chosen values of Reynolds number Re in the region of developed flow ($Re \in \langle 10^4, 10^6 \rangle$), has been solved numerically and the coefficients A, α, α_w have been found for which A_t is a minimum. The values of the friction factor have been determined on the basis of the Nikuradse equation [3]

$$\frac{1}{f^{1/2}} = 4.0\,\log(Re\,f^{1/2}) - 0.40. \quad (3.27)$$

The values of velocity U_0 at the center line have been determined for the respective values of Re from

$$\frac{1}{(U_0-1)^{1/4}} = 0.1078\,\log(Re\;U_0) + 0.9547. \quad (3.28)$$

These values have been combined over the interval $Re \in \langle 10^4, 10^6 \rangle$ with the experimental values found by Nikuradse [3]. The computed coefficients A, α, α_w are given in Table 1, together with other characteristic quantities, namely A_t, Y_{in}, $Y_{t\,max}$, $W_{t\,max}$, A_w, K. The coordinate of the inflexion point Y_{in} (situated nearer to the wall) of the eddy viscosity has been found by solving

$$\frac{d^2(\varepsilon/\nu)}{dY^2} = \frac{d^2H(Y)}{dY^2} = 0.$$

The approximate coordinates of the inflexion points for $\alpha > \alpha_w$ may be determined from

$$Y_{in\,1} \doteq (\tfrac{3}{2}\alpha_w)^{1/2}$$

and

$$Y_{in\,2} = (\tfrac{3}{2}\alpha)^{1/2}.$$

The coordinate of the turbulent energy density production maximum $Y_{t\,max}$ has been determined from the condition $dW_t/dY = 0$ and $W_{t\,max}$ from the equation (2.8d) for $Y = Y_{t\,max}$.

For the application, the following substitute relationships have been combined with the values of the coefficients for discrete values of the Reynolds number

$$A = 0.0029396(Re + 2000)^{0.9341}, \quad (3.29)$$

$$\alpha = 0.37252(Re - 9460)^{-0.01302}, \quad (3.30)$$

$$\alpha_w = 40198.7(Re - 1630)^{-1.7278}. \quad (3.31)$$

The coefficient A varies approximately linearly with Re. On the other hand the coefficient α is practically constant, while the coefficient α_w decreases rapidly with increasing Re.

In Table 1, some basic dimensionless parameters characteristic for turbulent flow are given, namely the dimensionless coefficient K^{++} expressing the proportionality of eddy viscosity near the wall to the third power of the dimensionless distance from the wall y^+,

$$K^{++} = \frac{K}{[(f/2)^{1/2}(Re/2)]^3}, \quad (3.32)$$

$$\frac{\varepsilon}{\nu} = K^{++}(y^+)^3, \quad (3.33)$$

$$y^+ = \left(\frac{f}{2}\right)^{1/2}\frac{Re}{2}; \quad Y = \left(\frac{\tau_w}{\rho}\right)^{1/2}\frac{y}{\nu}, \quad (3.34)$$

the dimensionless mean eddy viscosity,

$$\varepsilon_s^+ = A_t^+ = \frac{A_t}{[(f/2)^{1/2}(Re/2)]}, \quad (3.35)$$

the dimensionless distance of the inflex point of eddy viscosity from the wall,

$$y_{in}^+ = \left(\frac{f}{2}\right)^{1/2}\frac{Re}{2}\,Y_{in}, \quad (3.34a)$$

and the dimensionless distance of the turbulent energy density production maximum

$$y_{t\,max}^+ = \left(\frac{f}{2}\right)^{1/2}\frac{Re}{2}\,Y_{t\,max}. \quad (3.34b)$$

The coefficients $K^{++} \doteq 6 \times 10^{-4}$ and $\varepsilon_s^+ \doteq 0.06$ are approximately constant in the range of Re values which has been observed. The dimensionless distances $y_{t\,max}^+$

Table 1. Characteristics of turbulent flow in the pipe determined on the basis of the eddy viscosity model and experimental data

Re	f	U_0	$(f/4)Re$	$(f/2)^{1/2}(Re/2)$	A	α	α_w	A_t	A_w	K
1×10^{4}	0.772712×10^{-2}	1.2627	19.3128	3.1078×10^{2}	18.955151	0.34165449	0.66375×10^{-2}	18.586927	0.368251606	1.639247×10^{4}
1.5×10^{4}	0.695695×10^{-2}	1.2492	26.0886	4.4233×10^{2}	26.279065	0.33627352	0.2988×10^{-2}	26.045559	0.233505886	5.185032×10^{4}
2×10^{4}	0.647572×10^{-2}	1.2402	32.3750	5.6902×10^{2}	33,495600	0.33279743	0.1730×10^{-2}	33.321478	0.174122102	1.157521×10^{5}
3×10^{4}	0.587504×10^{-2}	1.2281	44.0628	8.1298×10^{2}	47.617780	0.32826558	0.8183×10^{-3}	47.499078	0.118701539	3.536506×10^{5}
4×10^{4}	0.549641×10^{-2}	1.2200	54.9641	1.0484×10^{3}	61.386696	0.32555432	0.4853×10^{-3}	61.295187	0.091508426	7.759346×10^{5}
5×10^{4}	0.522650×10^{-2}	1.2140	65.3313	1.2779×10^{3}	74.884943	0.32373417	0.3246×10^{-3}	74.809858	0.075085223	1.425236×10^{6}
6×10^{4}	0.501995×10^{-2}	1.2092	75.2993	1.5029×10^{3}	88.205239	0.32241636	0.2343×10^{-3}	88.141140	0.064098756	2.333569×10^{6}
8×10^{4}	0.471728×10^{-2}	1.2019	94.3456	1.9426×10^{3}	114.390630	0.32059393	0.1405×10^{-3}	114.34050	0.050131590	5.076921×10^{6}
1×10^{5}	0.450037×10^{-2}	1.1964	112.5090	2.3718×10^{3}	140.159750	0.31939622	0.9478×10^{-4}	140.11816	0.041592042	9.257159×10^{6}
1.5×10^{5}	0.414162×10^{-2}	1.1870	155.3107	3.4129×10^{3}	203.08206	0.31741108	0.4654×10^{-4}	203.05228	0.029776651	2.749087×10^{7}
2×10^{5}	0.391169×10^{-2}	1.1806	195.5845	4.4224×10^{3}	264.74642	0.31652462	0.2811×10^{-4}	264.72291	0.023511668	5.950546×10^{7}
3×10^{5}	0.361789×10^{-2}	1.1721	271.3418	6.3797×10^{3}	385.35837	0.31536810	0.1389×10^{-4}	385.34139	0.016972635	1.759364×10^{8}
4×10^{5}	0.342849×10^{-2}	1.1663	342.8490	8.2806×10^{3}	503.85686	0.31469645	0.8438×10^{-5}	503.84335	0.013509985	3.794847×10^{8}
5×10^{5}	0.329134×10^{-2}	1.1620	414.1850	1.0141×10^{4}	620.65108	0.31428700	0.5732×10^{-5}	620.63976	0.011319501	6.890284×10^{8}
6×10^{5}	0.318513×10^{-2}	1.1586	477.7695	1.1972×10^{4}	736.18290	0.31400756	0.4187×10^{-5}	736.17308	0.009816317	1.119877×10^{9}
8×10^{5}	0.302743×10^{-2}	1.1534	605.4860	1.5562×10^{4}	964.57789	0.31354173	0.2558×10^{-5}	964.57002	0.007869416	2.405303×10^{9}
1×10^{6}	0.291280×10^{-2}	1.1495	728.2025	1.9081×10^{4}	1190.3152	0.31329656	0.1740×10^{-5}	1190.3086	0.006610824	4.364528×10^{9}

Re	Y_{in}	$Y_{t\,max}$	$W_{t\,max}$	$Y_{\varepsilon=\nu}$	Y_{BL}	y^+_{in}	$y^+_{t\,max}$	$y^+_{\varepsilon=\nu}$	y^+_{BL}	A^+_t	K^{++}
1×10^{4}	0.09599619	0.040173382	0.2445	0.041066974	0.038896015	29.8344	12.4851	12.7628	12.0881	0.059807346	5.46102×10^{-4}
1.5×10^{4}	0.06570194	0.027585711	0.2460	0.028002225	0.027278800	28.8924	12.2020	12.3862	12.0662	0.058882642	6.02635×10^{-4}
2×10^{4}	0.05037152	0.021187387	0.2471	0.021431357	0.021208203	28.6625	12.0560	12.1948	12.0679	0.058559414	6.28268×10^{-4}
3×10^{4}	0.03484332	0.014654499	0.2480	0.014770407	0.014873871	28.3271	11.9138	12.0080	12.0922	0.058425887	6.58162×10^{-4}
4×10^{4}	0.02689167	0.011297073	0.2485	0.011365671	0.011563855	28.1950	11.8438	11.9158	12.1235	0.058455459	6.73310×10^{-4}
5×10^{4}	0.02200168	0.009236615	0.2487	0.009282343	0.009512756	28.1181	11.8035	11.8619	12.1564	0.058541246	6.82900×10^{-4}
6×10^{4}	0.01871678	0.007838715	0.2489	0.007871568	0.008110036	28.1312	11.7808	11.8302	12.1886	0.058647375	6.87390×10^{-4}
8×10^{4}	0.01450309	0.006053332	0.2491	0.006072856	0.006305236	28.1742	11.7592	11.7971	12.2485	0.058859518	6.92536×10^{-4}
1×10^{5}	0.01191566	0.004956249	0.2491	0.004969303	0.005186867	28.2616	11.7552	11.7862	12.3022	0.059076718	6.93811×10^{-4}
1.5×10^{5}	0.00835244	0.003448651	0.2495	0.003454945	0.003637685	28.5065	11.7699	11.7914	12.4151	0.059495526	6.91527×10^{-4}
2×10^{5}	0.00649221	0.002665860	0.2496	0.002669609	0.002828159	28.7117	11.7895	11.8061	12.5072	0.059859558	6.87977×10^{-4}
3×10^{5}	0.00456406	0.001856749	0.2497	0.001858559	0.001983463	29.1153	11.8455	11.8570	12.6539	0.060401177	6.77565×10^{-4}
4×10^{5}	0.00355743	0.001436511	0.2498	0.001437590	0.001542065	29.4580	11.8952	11.9041	12.7692	0.060846237	6.68357×10^{-4}
5×10^{5}	0.00293210	0.001177117	0.2498	0.001177840	0.001265846	29.7366	11.9371	11.9446	12.8643	0.061201041	6.70001×10^{-4}
6×10^{5}	0.00250601	0.001000862	0.2498	0.001001382	0.001081491	30.0021	11.9823	11.9885	12.9476	0.061491236	6.52620×10^{-4}
8×10^{5}	0.00195878	0.000775277	0.2499	0.000775586	0.000840817	30.4837	12.0649	12.0697	13.0848	0.061982394	6.38197×10^{-4}
1×10^{6}	0.00161554	0.000635236	0.2499	0.000635440	0.000691680	30.8269	12.1209	12.1248	13.1979	0.062381877	6.28236×10^{-4}

$\doteq 12$ and $y_{in}^+ \doteq 29$ are the same for all the values of Re which have been considered. The value $y_{t\,max}^+ = 12$ is in agreement with Laufer's experimental values [8] including the value $W_{t\,max}$ which differs only slightly from 0.25 for all Re values. The value $y_{in}^+ = 29$ corresponds to the boundary between the turbulent core and the buffer layer and is usually regarded to be in the range $y^+ = 27$–30 [3, 14], the universal dimensionless profile being expressed as $u^+ = f(y^+)$. The coordinate $Y_{t\,max}$ of the turbulent energy density production maximum is practically identical with the coordinate $Y_{\varepsilon=\nu}$ where the eddy viscosity ε equals the molecular viscosity ν, i.e. $\varepsilon/\nu = 1$, or where the local density of the turbulent production energy W_t equals the local density of the direct viscous dissipated energy W_ν. The coordinate $Y_{\varepsilon=\nu}$ which follows from the solution of equation (3.20) for $\varepsilon/\nu = 1$ and $y_{\varepsilon=\nu}^+$ are given in Table 1. The coordinates $Y_{t\,max}$ and $Y_{\varepsilon=\nu}$ differ only slightly from the values of Y_{BL} given by the Blasius relation [4]

$$Y_{BL} = \frac{123}{Re^{7/8}} \tag{3.36}$$

for the thickness of the viscous layer; the coordinates Y_{BL} and y_{BL}^+ are also given in Table 1.

When determining the values of the coefficients A, α and α_w using the experimental values of f and U_0 further conditions, such as the assumed minimum of the ratio of production of turbulent energy and direct viscous dissipated energy along the cross-section, were taken into account as well as the hypothesis of the mean eddy viscosity minimum. The computed velocity profiles determined for conditions other than the required mean eddy viscosity minimum along the pipe radius are not in agreement with the experimentally found profiles. It is therefore possible to regard the initial hypothesis as correct and to assume that momentum transport is not a cross-section function but a path function.

The procedure for determining the coefficients A, α and α_w may be modified, for example using the fact that $y_{in}^+ = $ const. instead of the experimental values of the velocity U_0.

The fact that the dimensionless coordinates y_{in}^+ and $y_{t\,max}^+$ are practically independent of the flow regime (i.e. of Reynolds number Re) leads to a consideration of the inner connections of the turbulent fluid flow in these quantities. From the hydrodynamic equations only such dependencies follow that allow the interpretation of the physical meaning of these variables. No relationship gives a quantitative expression of any linkage. This is also why the statistical characteristics of the eddy viscosity course (so called 'quantiles') have been analyzed and a connection between the coordinate of energy production turbulent maximum and the values of the coefficients A and A_w or α and α_w have been found. It has been found that the coordinate $Y_{t\,max}$ and the coordinates of the point where the sum of the integral effect of both the components of the eddy viscosity is equal to one half of the total effect of the component near the wall coincide, i.e.

$$\int_0^{Y_{t\,max}} \frac{2A}{\alpha} G(Y)\,\mathrm{d}Y + \int_0^{Y_{t\,max}} \frac{2A_w}{\alpha_w} G_w(Y)\,\mathrm{d}Y = \frac{A_w}{2}. \tag{3.37}$$

After simplifying this equation by leaving out of the unessential terms near the wall which contain the coordinates $(2-Y)$, $(2+Y)$, $(4-Y)$, replacing $A_w/A = \alpha_w/\alpha$, and after integration we obtain

$$\alpha(1-e^{-Y_{t\,max}^2/\alpha}) + \alpha_w(1-e^{-Y_{t\,max}^2/\alpha_w}) = \frac{\alpha_w}{2}. \tag{3.37a}$$

The deviations between $Y_{t\,max}$ determined on the basis of equations (2.8d) and (3.37a) amount, in the observed interval $Re \in \langle 10^4, 10^6 \rangle$, to 0.14% for small Reynolds numbers and to 0.000002% for large Re.

The finding that the coefficient α varies only very little with Re and that the mean value of reach of the wall influence η ($\alpha_{max} = 0.341655 \Rightarrow \eta_{max} = 0.5180$, $\alpha_{min} = 0.313297 \Rightarrow \eta_{min} = 0.4998$), determined on the basis of this coefficient from the equation (3.11), varies even less, leads to the consideration that the mean value of the wall influence reach into the turbulent core equals practically one half of the pipe radius, i.e. $\eta = 0.5$, with which is associated, according to equation (3.11), the coefficient $\alpha = 1/\pi$, which is constant throughout the region of the developed flow. This value of the coefficient α is associated with the probability of the reach of the wall influence towards the opposite wall ($Y = 2$),

$$P(2) = 1 - e^{2^2\pi} = 1 - 3.487 \times 10^{-6} = .999996513.$$

The finding of the connection between the coefficients A and A_w, or α and α_w and the coordinate $Y_{t\,max}$ of the turbulent energy production maximum, expressed in equation (3.37a), together with the simplification $\alpha = 1/\pi = $ const. for all the regimes, enables the methodology of adjoining the coefficients A, A_w and α_w to the respective flow regimes to be simplified to a procedure which does not require the extreme of the coefficient A_t to be found. For the chosen value of the coefficient α_w, if $\alpha = 1/\pi$ is considered, the coordinate $Y_{t\,max}$ will be found from equation (3.37a). When the derivative of equation (2.8d) is put equal to zero, and the coordinate $Y = Y_{t\,max}$ determined from equation (3.37a) is inserted, the following expression for the coefficient A, or $2A/\alpha = 2A_w/\alpha_w$ will be obtained:

$$\frac{2A}{\alpha} = \frac{\frac{\mathrm{d}H(Y)}{\mathrm{d}Y}(1-Y) - 2H(Y)}{\frac{\mathrm{d}H(Y)}{\mathrm{d}Y}H(Y)(1-Y) + 2[H(Y)]^2}. \tag{3.38}$$

For these three coefficients the corresponding flow regime will be determined by way of a numerical computation from equation (2.6a), [the complex $(f/4)Re$]; the other quantities will be determined from their respective equations, e.g. Y_{in}, $W_{t\,max}$, A_t, A_w, σ_w^2, σ_w, K, the values of which, for $\alpha_w = 0.01$, 0.001, 0.0001, 0.00001, 0.000001, are given in Table 2. From equation

Table 2. Characteristics of turbulent flow in the pipe determined on the basis of the eddy viscosity model

α_w	10^{-6}	10^{-5}	10^{-4}	10^{-3}	10^{-2}
$Y_{t\,max}$	0.00051595685	0.0016316927	0.0051599626	0.016319054	0.051660464
$W_{t\,max}$	0.24974205	0.24918446	0.24742313	0.24187166	0.22448007
A	1317.8547	416.01126	130.91019	40.833476	12.583644
A_t	1317.8505	415.99819	130.86907	40.705194	12.188317
A_w	0.004140163	0.01306979	0.041126649	0.128282148	0.395326836
Y_{in}	0.0012247362	0.0038727110	0.012238858	0.038464175	0.11547985
α	—	—	0.3183098861	—	—
η	—	—	0.5	—	—
K	8.28029×10^9	2.61379×10^8	8.22274×10^6	2.55758×10^5	7.65814×10^3
U_0	1.1536731	1.1740900	1.2010321	1.2407182	1.3100513
$(f/4)Re$	828.88808	296.48076	107.66181	40.019964	15.537811
f	0.0028363605	0.0035443052	0.0045531461	0.0060528858	0.0083737127
Re	1168946.0	334588.29	94582.349	26446.865	7422.1140
A_t^+	0.059873672	0.059069072	0.057998414	0.055955061	0.050757473
$y^+_{t\,max}$	11.356475	11.491314	11.643068	11.871496	12.40515
y^+_{in}	26.957073	27.273846	27.616065	27.981235	27.730006
K^{++}	0.000776525	0.000748302	0.000715738	0.000664352	0.000553085
R_{max}	0.99870916	0.99736798	0.99390987	0.98502623	0.96228725
$(q/q_w)_{max}$	1.0003	1.0009	1.0025	1.0068	1.0178
R_1	0.99584818	0.99180365	0.98278685	0.96177780	0.911674050

(2.5), in the integration interval $\langle 0, 1 \rangle$, the velocity at the center line U_0 was fixed.

From the complex $(f/4)Re$, with the use of Nikuradse relation (3.27) for the Fanning friction factor, both the friction factor f corresponding to the chosen value α_w and the Reynolds number Re have been determined. The values are given in Table 2.

From the discrete values of the dependence between α_w and Re, the equivalent analytic dependence has been found,

$$\alpha_w = \frac{109255.5}{Re^{1.8175}}, \tag{3.39}$$

which after certain adaptation gives

$$Re = \frac{591.76}{\alpha_w^{0.5502}}. \tag{3.39a}$$

With the aid of the friction velocity $(\tau_w/\rho)^{1/2} = (f/2)^{1/2}u_s$, the dimensionless coordinates of the inflexion point of eddy viscosity y^+_{in} have been expressed as well as the dimensionless coordinates of the turbulent energy density maximum $y^+_{t\,max}$, which are given, together with the dimensionless coefficient K^{++} in Table 2.

In Fig. 1 the total relative viscosity $(\nu+\varepsilon)/\nu$ along the pipe radius is represented, computed on the basis of the model for several values of α_w. ε^+ near the wall, showing the proportionality of the eddy viscosity to the third power of the distance from the wall Y, is plotted in Fig. 2. The course of the velocity U near the wall, where it is practically linear, is represented in Fig. 3; the velocity profiles expressed in dimensionless form $u^+ = f(y^+)$ are given for the basic values of α_w in Fig. 4.

A comparison of the complex $(f/4)Re$ dependence on the coefficient α_w, according to the model of eddy viscosity (full line) and the values obtained from Nikuradse data for the center line velocity and the

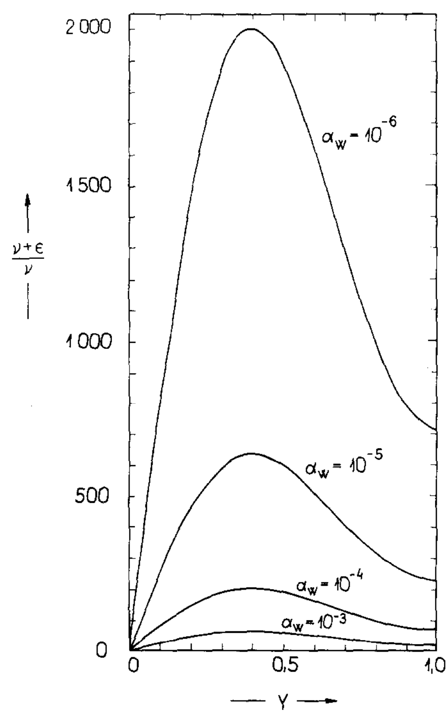

FIG. 1. Total relative viscosity $(\nu+\varepsilon)/\nu = f(Y, \alpha_w)$.

friction factor (dotted line), is shown in Fig. 5. For large values of α_w the values of $(f/4)Re$ are given, obtained from the data [16] (open circles). In Fig. 6 there is a comparison of the computed values of the center line velocity U_0 and of the experimental data for the identical α_w.

In thermokinetic calculations for q_w = const., the ratio of local heat flux density to heat flux density on the pipe wall, q/q_w, occurs. This depends on the hydraulic

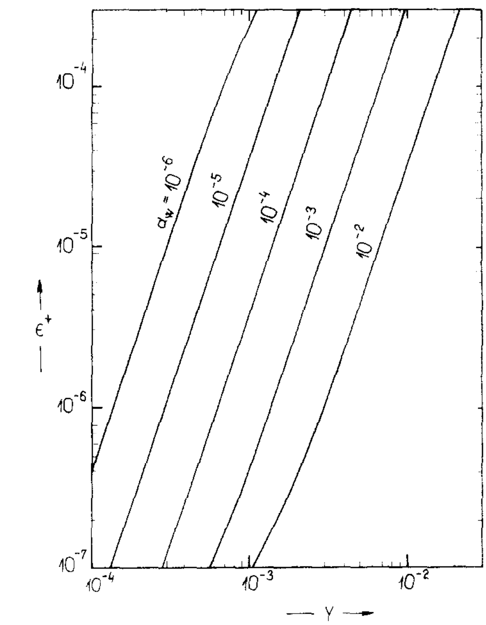

FIG. 2. Dimensionless eddy viscosity near the wall $\varepsilon^+ = f(Y, \alpha_w)$.

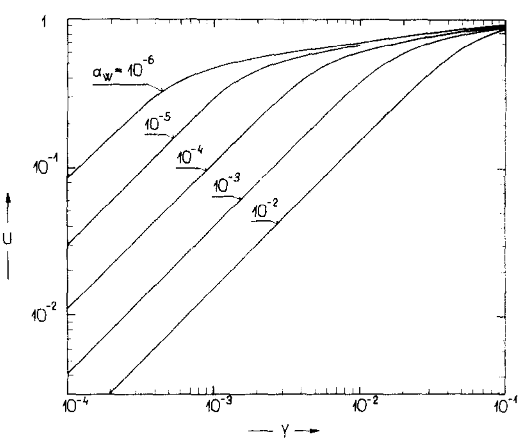

FIG. 3. Velocity profiles near the wall.

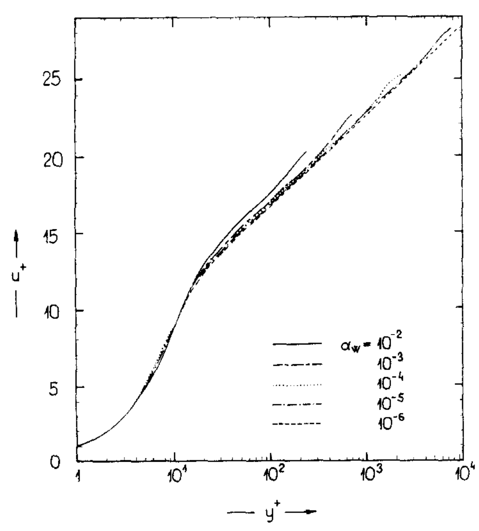

FIG. 4. Dimensionless velocity profiles $u^+ = f(y^+, \alpha_w)$.

parameters of the flow only, as is clearly shown by the relation

$$\frac{q}{q_w} = \frac{2}{R}\int_0^R UR\,dR. \tag{3.40}$$

For the basic values of α_w the ratio q/q_w is represented in Fig. 7, as its dependence on the coordinate $Y = 1 - R$; Figure 8 shows the same dependence in semi-logarithmic coordinates. The radius R_{max}, at which q/q_w reaches the maximum, and which is determined from the condition $d(q/q_w)/dR = 0$ leading to

$$UR_{max}^2 = \int_0^{R_{max}} UR\,dR, \tag{3.41}$$

is given in Table 2 together with the values of $(q/q_w)_{max}$ and the values of the radius R_1, where $q/q_w = 1$. Figure 7 also shows the relative Reynolds turbulent shear stress τ_t/τ_w along the pipe radius. Figure 9 shows the same dependence in semilogarithmic coordinates for the basic values of the coefficients α_w.

The Reynolds turbulent shear stress expressed by the ratio τ_t/τ_w has been determined from

$$\frac{\tau_t}{\tau_w} = \frac{\overline{u'v'}}{u^{*2}} = R\left(1 - \frac{1}{(v+\varepsilon)/v}\right)$$

$$= (1-Y)\frac{\varepsilon/v}{(v+\varepsilon)/v}. \tag{3.42}$$

The coordinate $Y_{\tau\,max}$ of the Reynolds turbulent shear stress maximum is determined by the condition of the extreme $d(\tau_t/\tau_w)/dY = 0$

$$1 + \frac{2A}{\alpha}H(Y) + (1-Y)\frac{2A}{\alpha}H'(Y) - \left(\frac{2A}{\alpha}\right)^2[H(Y)]^2 = 0. \tag{3.43}$$

The course of the turbulent energy production rate W_t and the direct viscous dissipated energy rate W_v, as functions of y^+, are shown in Fig. 10, together with the values of these quantities obtained in an experiment by Laufer [8] for $Re\,U_0 = 5 \times 10^4$ and 5×10^5.

Figure 11 shows the dependence of the coefficient K on the Reynolds number Re compared with data obtained by other authors.

4. THE BASIC EQUATIONS OF THE TURBULENT HEAT TRANSFER

Convective heat transport in turbulent flow is described by the differential equation [14]

$$\frac{1}{r}\frac{\partial}{\partial r}\left[(\lambda + \rho c_p \varepsilon_q) r \frac{\partial T}{\partial r}\right] = \rho c_p u \frac{\partial T}{\partial x}. \tag{4.1}$$

Let us limit the problem to the solution of the case of the constant heat flux density on the wall $q_w = \text{const.}$

$$q_w = h(T_w - T_s) = \text{const.} \tag{4.2}$$

If the relative temperature $\Theta = (T_w - T)/(T_w - T_s)$ is

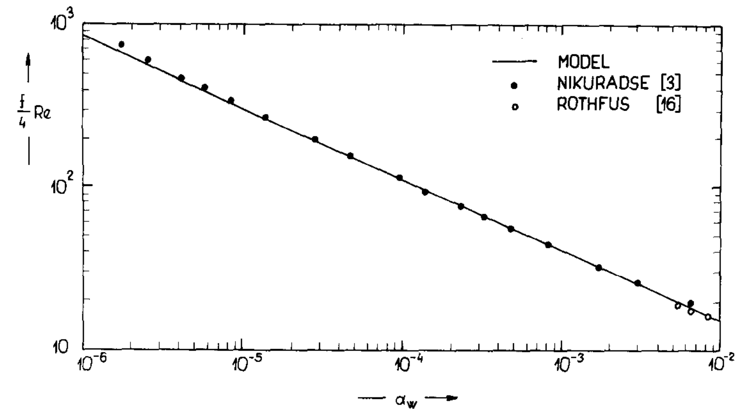

FIG. 5. Complex $(f/4)Re = f(\alpha_w)$.

applied where

$$T_w - T_s = \frac{2}{u_s r_w^2} \int_0^{r_w} u(T_w - T) r \, dr \tag{4.3}$$

for

$$\frac{\partial T}{\partial x} = \frac{\partial T_s}{\partial x} = \frac{\partial T_w}{\partial x} = \text{const.}$$

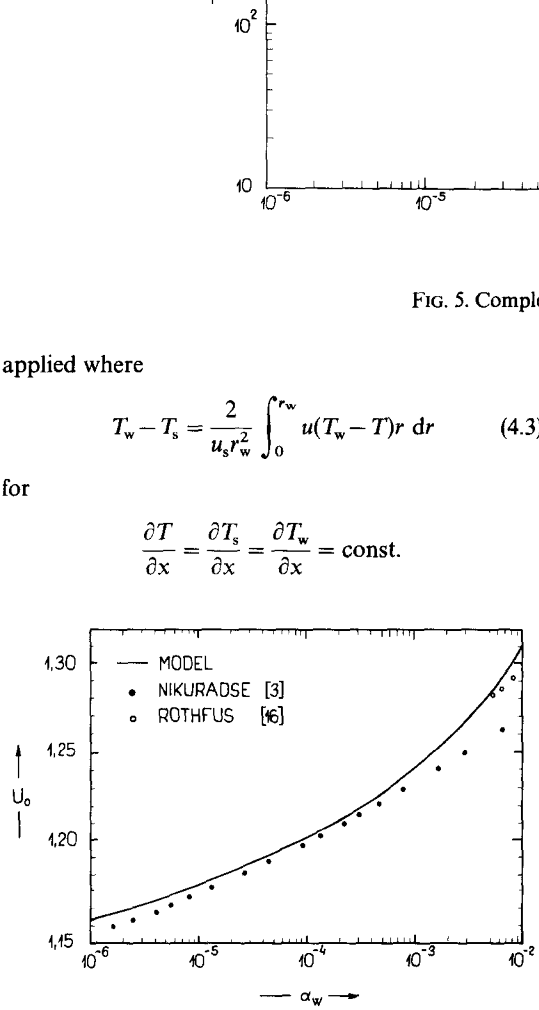

FIG. 6. Relative center line velocity $U_0 = f(\alpha_w)$.

the initial equation may be transformed into a dimensionless form

$$\frac{1}{R}\frac{d}{dR}\left[\frac{a+\varepsilon_q}{a} R \frac{d\Theta}{dR}\right] = -NuU \tag{4.1a}$$

and the defining equation of the medium bulk temperature (4.3) may be changed into the form

$$\int_0^1 \Theta U R \, dR = \frac{1}{2}. \tag{4.3a}$$

For the boundary conditions $R = 0 \Rightarrow d\Theta/dR = 0$; $R = 1 \Rightarrow \Theta = 0$, after a double formal integration of equation (4.1a), the following expression for temperature will be obtained

$$\Theta = Nu \int_R^1 \frac{\int_0^R UR \, dR}{[R(a+\varepsilon_q)/a]} dR. \tag{4.4}$$

After inserting the obtained expression into the normalizing condition (4.3a) corresponding to the dimensionless equation of energy conservation, the following relation for the Nusselt number will be

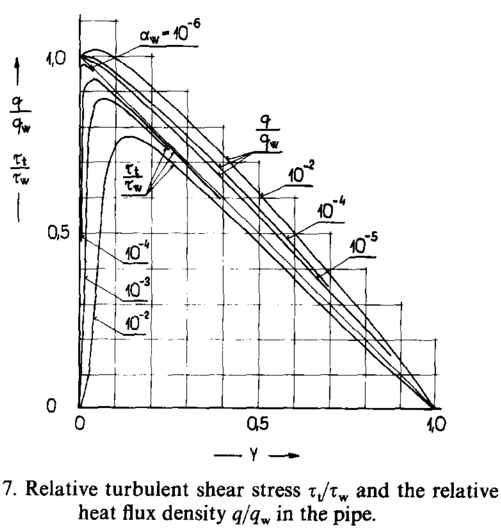

FIG. 7. Relative turbulent shear stress τ_t/τ_w and the relative heat flux density q/q_w in the pipe.

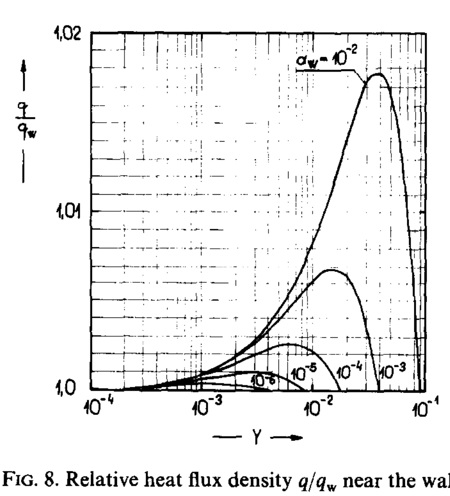

FIG. 8. Relative heat flux density q/q_w near the wall.

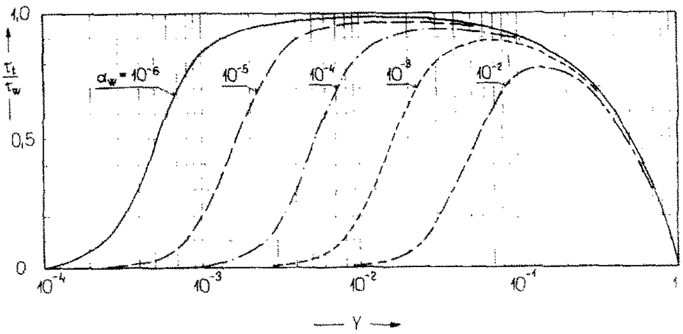

FIG. 9. Relative turbulent shear stress τ_t/τ_w near the wall.

obtained:

$$Nu = \frac{1}{2\int_0^1 UR \int_R^1 \left\{\int_0^R UR\,\mathrm{d}R/[R(a+\varepsilon_q)/a]\right\}\mathrm{d}R} \tag{4.5}$$

which, when the sequence of integration is interchanged, may be rewritten into the equivalent form

$$Nu = \frac{1}{2\int_0^1 \left\{\left(\int_0^R UR\,\mathrm{d}R\right)^2/[R(a+\varepsilon_q)/a]\right\}\mathrm{d}R}, \tag{4.5a}$$

known in the literature as the Lyon relation [9].

Further, it follows from the boundary conditions for the relative temperature gradient on the wall that

$$\left(\frac{\mathrm{d}\Theta}{\mathrm{d}R}\right)_w = -\frac{Nu}{2}. \tag{4.6}$$

The relations valid for laminar flow follow directly from the above stated relations for turbulent flow when $\varepsilon_q/a \equiv 0$ is inserted. By integration of the differential equation for the turbulent heat transport (4.1) in the limit of $\langle 0, r\rangle$, and after multiplying by the temperature gradient, an equation analogical to the energy equation

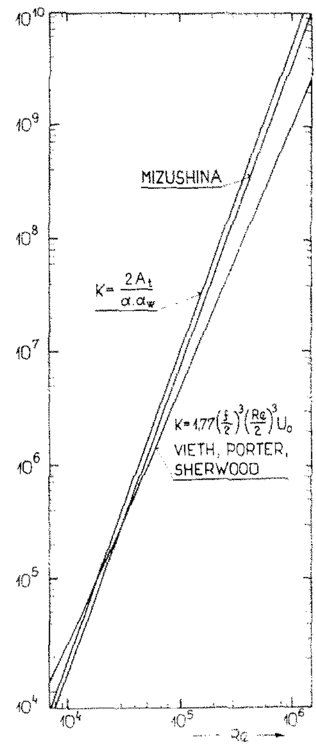

FIG. 11. Coefficient K in equation (3.22).

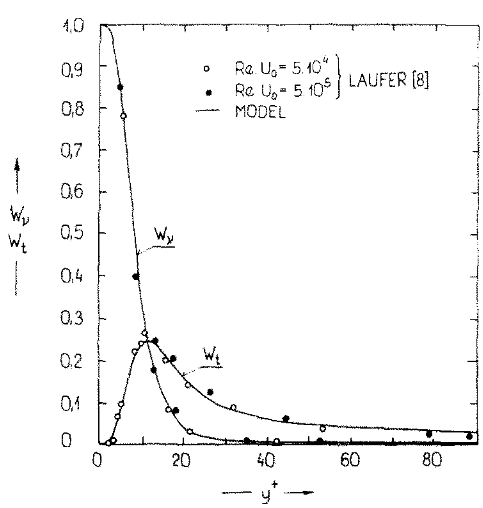

FIG. 10. Course of the production turbulent energy rate W_t and the direct viscous dissipated energy rate W_ν near the wall.

(2.8a) for the turbulent momentum transport will be obtained which, when expressed dimensionlessly, has the form

$$\frac{4\left(\int_0^R UR\,\mathrm{d}R\right)^2}{R^2[(a+\varepsilon_q)/a]^2} + \frac{\varepsilon_q}{a}\,\frac{4\left(\int_0^R UR\,\mathrm{d}R\right)^2}{R^2[(a+\varepsilon_q)/a]^2} = \frac{4\left(\int_0^R UR\,\mathrm{d}R\right)^2}{R^2[(a+\varepsilon_q)/a]}. \tag{4.7}$$

When q/q_w from equation (3.40) is inserted an equivalent expression is obtained

$$\frac{(q/q_w)^2}{[(a+\varepsilon_q)/a]^2} + \frac{\varepsilon_q}{a}\,\frac{(q/q_w)^2}{[(a+\varepsilon_q)/a]^2} = \frac{(q/q_w)^2}{[(a+\varepsilon_q)/a]}, \tag{4.7a}$$

$$W_{qa} + W_{qt} = W_q. \tag{4.7b}$$

The first term on the LHS, W_{qa}, has a meaning analogical for heat transport with the direct viscous dissipated energy rate in momentum transport, i.e. the relative density of molecularly transported energy; the second term W_{qt} corresponds to the relative density of turbulently transported energy. The total W_q expresses the relative density of transported thermal energy; it follows from a comparison of equations (4.7a) and (4.7b) that

$$\frac{\varepsilon_q}{a} = \frac{W_{qt}}{W_{qa}}. \quad (4.8)$$

The mean value of $(\varepsilon_q/a)_s$ along the pipe radius equals the ratio of the integral values of turbulent and molecular energy densities,

$$\left(\frac{\varepsilon_q}{a}\right)_s = \int_0^1 \frac{W_{qt}}{W_{qa}}\,\mathrm{d}R = \int_0^1 \frac{\varepsilon_q}{a}\,\mathrm{d}R. \quad (4.9)$$

The mean value of eddy diffusivity of heat and the mean value of eddy viscosity also enable the mean value of the turbulent Prandtl number along the pipe radius to be determined,

$$Pr_{ts} = \frac{\varepsilon_s}{\varepsilon_{qs}} = \frac{(\varepsilon/\nu)_s}{(\varepsilon_q/a)_s}\,Pr. \quad (4.10)$$

The contribution of turbulent heat transport, namely the relative turbulent heat flux density is given by

$$\frac{q_t}{q_w} = \frac{\overline{v'T'}}{u^*\Delta T^*} = \frac{\overline{v'T'}\rho c_p}{q_w} = \frac{q}{q_w}\left[1 - \frac{1}{(a+\varepsilon_q)/a}\right] = \frac{q}{q_w}\,\frac{\varepsilon_q/a}{(a+\varepsilon_q)/a}. \quad (4.11)$$

By inserting q/q_w from equation (3.40) and W_q from equation (4.7b) into equations (4.5a) and (4.4) for Nusselt number Nu and for the relative temperature Θ, the following expression explaining the meaning of these quantities is obtained:

$$\frac{Nu}{2} = \frac{1}{\int_0^1 W_q R\,\mathrm{d}R}, \quad (4.5b)$$

$$\Theta = \frac{Nu}{2}\int_R^1 W_{qa}^{1/2}\,\mathrm{d}R = \frac{Nu}{2}\int_R^1 \frac{W_q}{(q/q_w)}\,\mathrm{d}R. \quad (4.4a)$$

From the above stated facts in comparison with the relations for $(f/4)Re$ and U_0 an analogy follows showing that the quantity equivalent to shear stress τ/τ_w in the case of momentum transport corresponds to the relative density of the transverse heat flux q/q_w in heat transfer. For the hydraulic resistances the decisive quantity is the total energy rate W; for heat transport the decisive quantity is the total absorbed energy density distribution W_q.

5. MODEL OF THE EDDY DIFFUSIVITY OF HEAT

As in the case of the eddy viscosity model, our considerations will be based on the idea of the origin of the eddy diffusivity of heat resulting from two stochastic processes (acting reversely) governed by the Rayleigh probability density distribution

$$f_q(Y) = \frac{2}{\alpha_q}\,Y\,\mathrm{e}^{-Y^2/\alpha_q}, \quad (5.1a)$$

$$f_{qw} = \frac{2}{\alpha_{qw}}\,Y\,\mathrm{e}^{-Y^2/\alpha_{qw}}. \quad (5.1b)$$

By superposition (difference) of these processes the condition of the proportionality of the eddy diffusivity of heat to the third power of the distance from the wall (in close proximity to the wall), which is valid for fluids with low Prandtl numbers, may be satisfied

$$\frac{\varepsilon_q}{a} \sim A_q f_q(Y) - A_{qw} f_{qw}(Y). \quad (5.2)$$

The proportionality to the fourth power of the distance, found for fluids with high Prandtl numbers, satisfies the superposition of the processes with a more general Weibull distribution of the probability density

$$f_q(Y) = \frac{\beta}{\alpha_q}\,Y^{\beta-1}\mathrm{e}^{-Y^\beta/\alpha_q} \quad (5.1c)$$

$$f_{qw}(Y) = \frac{\beta}{\alpha_{qw}}\,Y^{\beta-1}\,\mathrm{e}^{-Y^\beta/\alpha_{qw}} \quad (5.1d)$$

for the value of the coefficient $\beta = \frac{5}{2}$.

Let us limit our considerations to fluids with low Prandtl numbers and take for the starting point the Rayleigh probability density distribution. It is then possible to write for the eddy diffusivity of heat an equation formally identical with equation (3.20) for eddy viscosity which differs only in the values of the respective coefficients

$$\frac{\varepsilon_q}{a} = \frac{2A_q}{\alpha_q}\left[Y\mathrm{e}^{-Y^2/\alpha_q} + (2-Y)\mathrm{e}^{-(2-Y)^2/\alpha_q} - (2+Y)\mathrm{e}^{-(2+Y)^2/\alpha_q} - (4-Y)\mathrm{e}^{-(4-Y)^2/\alpha_q}\right]$$
$$- \frac{2A_{qw}}{\alpha_{qw}}\left[Y\mathrm{e}^{-Y^2/\alpha_{qw}} + (2-Y)\mathrm{e}^{-(2-Y)^2/\alpha_{qw}} - (2+Y)\mathrm{e}^{-(2+Y)^2/\alpha_{qw}} - (4-Y)\mathrm{e}^{-(4-Y)^2/\alpha_{qw}}\right]. \quad (5.3)$$

If the eddy diffusivity of heat near the wall is expressed by

$$\frac{\varepsilon_q}{a} = K_q Y^3 \quad (5.4)$$

the coefficient K_q is linked with the coefficients A_q, A_{qw},

α_q, α_{qw} by the equation

$$K_q = \frac{2(A_q - A_{qw})}{\alpha_q \alpha_{qw}} = \frac{2A_{qt}}{\alpha_q \alpha_{qw}} = \frac{2A_q}{\alpha_q \alpha_{qw}}\left(1 - \frac{\alpha_{qw}}{\alpha_q}\right), \qquad (5.5)$$

the ratio of the coefficients being given by

$$\frac{A_q}{\alpha_q} = \frac{A_{qw}}{\alpha_{qw}} \qquad (5.6)$$

which follows from the condition $[\mathrm{d}(\varepsilon_q/a)/\mathrm{d}Y]_w = 0$. The coefficient A_{qt} is the mean value of the relative eddy diffusivity of heat along the pipe radius

$$A_{qt} = A_q - A_{qw} = A_q\left(1 - \frac{\alpha_{qw}}{\alpha_q}\right) = \left(\frac{\varepsilon_q}{a}\right)_s = \int_0^1 \frac{\varepsilon_q}{a}\,\mathrm{d}Y. \qquad (5.7)$$

It is possible to use for the determination of the extreme and inflexion points or other quantities affecting the eddy diffusivity of heat the expressions given for eddy viscosity in Section 3.

When determining the coefficients from equation (5.3) for the eddy diffusivity of heat it is assumed that, as in the case of eddy viscosity, the distribution of probability density of the basic process influence is determined, for fully developed convection, by the pipe dimensions. Therefore,

$$\alpha_q = \alpha = \frac{1}{\pi}. \qquad (5.8)$$

Classical fluids

In accordance with Corrsin's finding of proportionality between the ratio of temperature and velocity microscales and the value $Pr^{1/2}$ which was derived for isotropic turbulence [12], let

$$A_q = APr^{1/2}. \qquad (5.9)$$

For $\alpha_q = \alpha$, it then follows from equation (5.6) that

$$\frac{\alpha_{qw}}{A_{qw}} = \frac{\alpha_w}{Pr^{1/2}A_w}. \qquad (5.10)$$

As the thickness of the thermal sublayer in the region near the wall decreases with increasing Prandtl number [5, 14], it follows from equations (5.10) and (3.23) that

$$\alpha_{qw} = \frac{\alpha_w}{Pr^{1/2}}, \qquad (5.11)$$

$$A_{qw} = A_w. \qquad (5.12)$$

For stochastic processes there are dispersions [11]

$$\sigma^2_{Yq} = \frac{4-\pi}{4}\alpha_q = \sigma^2_Y, \qquad (5.13a)$$

$$\sigma^2_{Yqw} = \frac{4-\pi}{4}\alpha_{qw} = \frac{4-\pi}{4}\frac{\alpha_w}{Pr^{1/2}} = \frac{\sigma^2_{Yw}}{Pr^{1/2}}. \qquad (5.13b)$$

Liquid metals

The model of the eddy diffusivity of heat in classical fluids above may be extended to and modified for liquid metals which have low Prandtl numbers $Pr \in \langle 0.003, 0.1 \rangle$ [5].

To determine the velocity profiles of liquid metals which are included in the category of Newtonian fluids the model of eddy viscosity may be applied without any changes.

The Prandtl number occurs in both the free and forced convection of liquid metals raised to twice the power in comparison with the classical fluids [5, 23, 24]. This fact allows a simple modification of the eddy diffusivity model to be made. In the relation for the eddy diffusivity of heat of liquid metals in the region of fully developed convection coefficients occur which are linked with the values of the coefficients in equation (3.20) for the eddy viscosity by

$$A_q = APr, \qquad (5.14)$$

and

$$\alpha_{qw} = \frac{\alpha_w}{Pr}. \qquad (5.15)$$

As for classical fluids we will put

$$\alpha_q = \alpha\left(= \frac{1}{\pi}\right). \qquad (5.8)$$

For the stochastic processes in liquid metals there are the respective dispersions

$$\sigma^2_{Yq} = \sigma^2_Y, \qquad (5.16a)$$

$$\sigma^2_{Yqw} = \frac{\sigma^2_{Yw}}{Pr}. \qquad (5.16b)$$

For a few hydraulic regimes defined by the value α_w the values Nu and Θ_0 have been computed for fluids with $Pr = 0.72$, 3 and 10. A comparison of the obtained dependencies of Nu on Reynolds number with those according to the relation [25]

$$Nu = 0.023\,Re^{0.8}\,Pr^n \qquad (5.17)$$

for $n = 0.33$ and $n = 0.4$ is given in Fig. 12.

The values of the relative center line temperatures Θ_0 for fluids identical in dependence on Re are given in Fig. 13. The linear temperature profiles near the wall, computed on the basis of the model, are shown in Fig. 14. For $Re = 94\,582$ ($\alpha_w = 0.0001$) the computed dimensionless temperature profiles $T^+ = f(y^{++})$ are given in Fig. 15.

In Fig. 16 the dimensionless temperature profiles for $Pr = 0.72$ are shown. The obtained temperature profiles show complete agreement with the data found in the literature [14, 3]. Plots of $(a+\varepsilon_q)/a$ for $Pr = 0.72$, 3 and 10 are shown in Fig. 17.

The dimensionless eddy diffusivity of heat,

$$\varepsilon_q^+ = \frac{\varepsilon_q}{u^* r_w} = \frac{\varepsilon_q}{a}\frac{1}{[(f/2)^{1/2}(Re/2)Pr]} \qquad (5.18)$$

shown in Fig. 18 to depend on log Y, confirms the proportionality of the eddy diffusivity of heat to the third power of the distance from the wall [equation (5.3)].

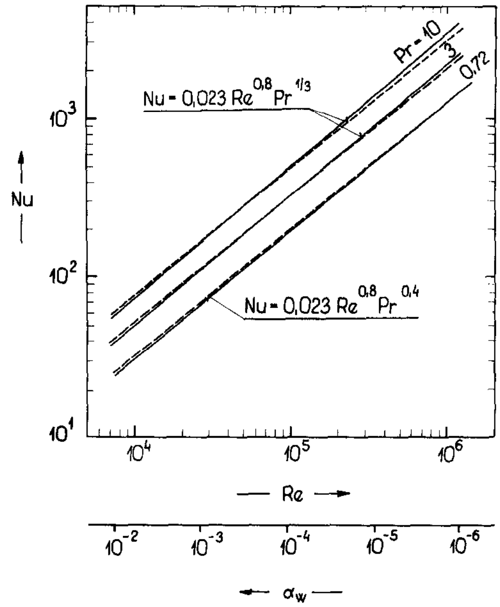

FIG. 12. Nusselt number $Nu = f(Re, Pr)$.

The turbulent contribution to the transverse heat flux along the pipe radius expressed by the ratio q_t/q_w according to the equation (4.11) is shown for several regimes with $Pr = 0.72$ in Fig. 19. In Fig. 20 the relative densities of molecularly- W_{qa} and turbulently-absorbed energy W_{qt} are shown as functions of the dimensionless coordinate y^+ for fluids with Prandtl number $Pr = 0.72$, for regimes corresponding to $\alpha_w \in \langle 0.001; 0.000001 \rangle$. The course of the local turbulent Prandtl number $Pr_t = \varepsilon/\varepsilon_q$ along the pipe radius is given for the regime corresponding to $\alpha_w = 10^{-4}$ for fluids with

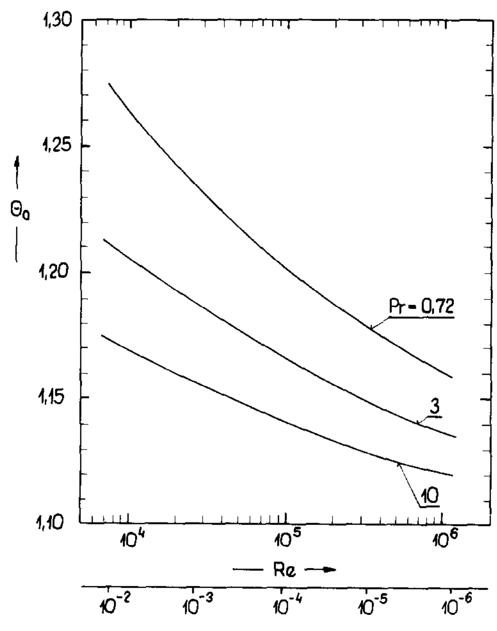

FIG. 13. Relative center line temperature $\Theta_0 = f(Re, Pr)$.

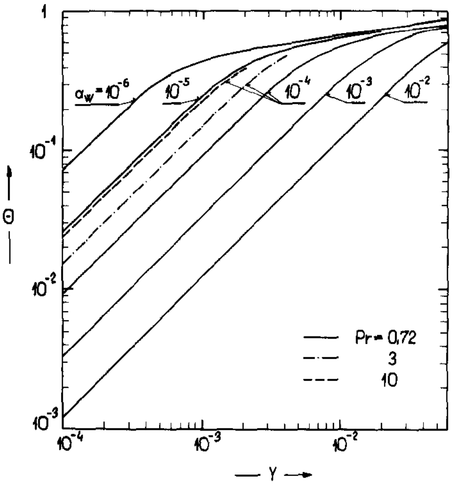

FIG. 14. Temperature profiles near the wall.

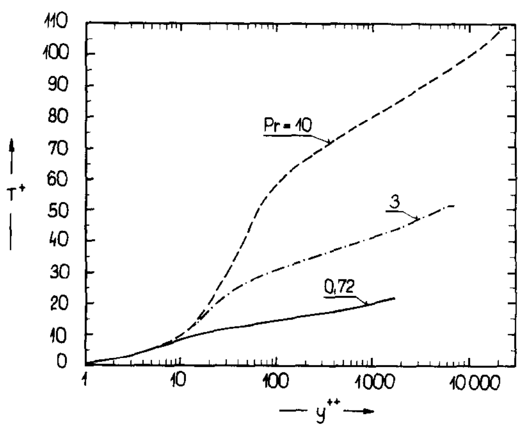

FIG. 15. Dimensionless temperature profiles $T^+ = f(y^{++}, Pr)$, $Re = 94\,582$.

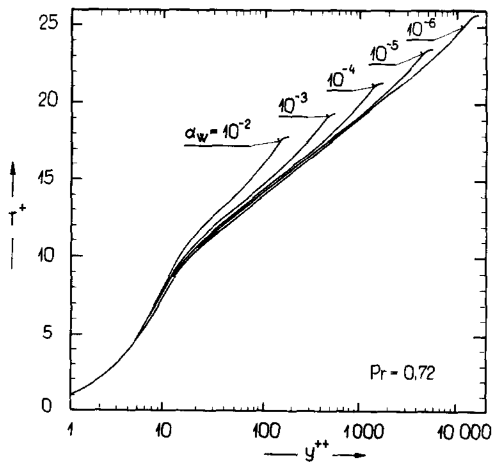

FIG. 16. Dimensionless temperature profiles $T^+ = f(y^{++}, \alpha_w)$, $Pr = 0.72$.

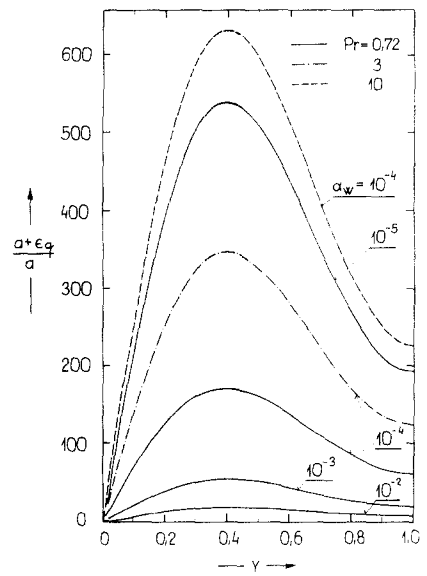

FIG. 17. Total relative diffusivity of heat $(a+\varepsilon_q)/a = f(Y)$.

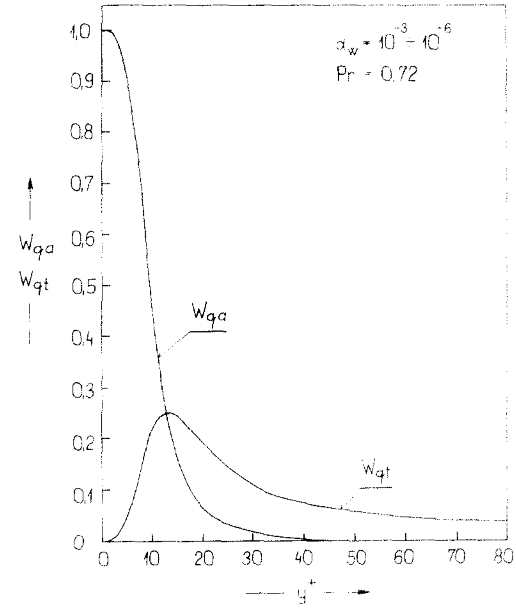

FIG. 20. Relative energy absorbed molecularly W_{qa} and turbulently W_{qt} near the wall.

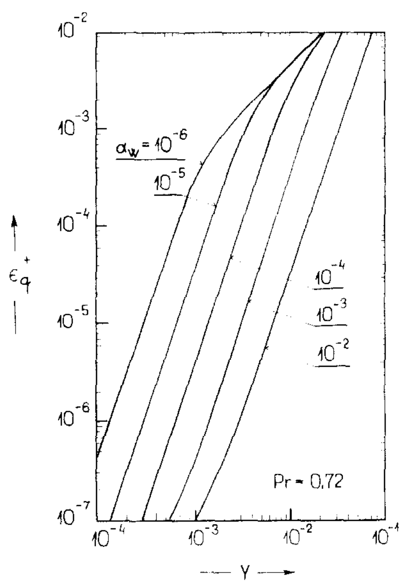

FIG. 18. Dimensionless eddy diffusivity of heat near the wall $\varepsilon_q^+ = f(Y, \alpha_w)$, $Pr = 0.72$.

Prandtl number $Pr = 0.72, 3$ and 10 in Fig. 21. In Fig. 22 Pr_t values for fluids with $Pr = 0.72$ for various regimes are shown. The value of the local turbulent Prandtl number on the wall Pr_{tw} is given by

$$Pr_{tw} = \frac{K}{K_q} Pr = \frac{A_t \alpha_q \alpha_{qw}}{A_{qt} \alpha \alpha_w} Pr = Pr_{ts} \frac{\alpha_q \alpha_{qw}}{\alpha \alpha_w}, \quad (5.19)$$

which for $\alpha_q = \alpha$ and $\alpha_{qw} = \alpha_w/(Pr^{1/2})$ is simplified to

$$Pr_{tw} = \frac{Pr_{ts}}{Pr^{1/2}}. \quad (5.19a)$$

The coordinates of the extreme value of the local turbulent Prandtl number may be determined from the following condition [equations (3.20) and (5.3)]:

$$\frac{\mathrm{d}Pr_t}{\mathrm{d}Y} = \frac{\mathrm{d}[(\varepsilon/\nu)/(\varepsilon_q/a)]}{\mathrm{d}Y} Pr = 0 \quad (5.20)$$

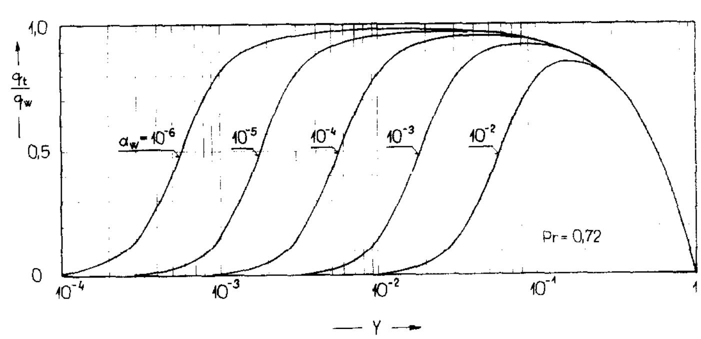

FIG. 19. Relative density of turbulent heat flux $q_t/q_w = f(Y, \alpha_w)$.

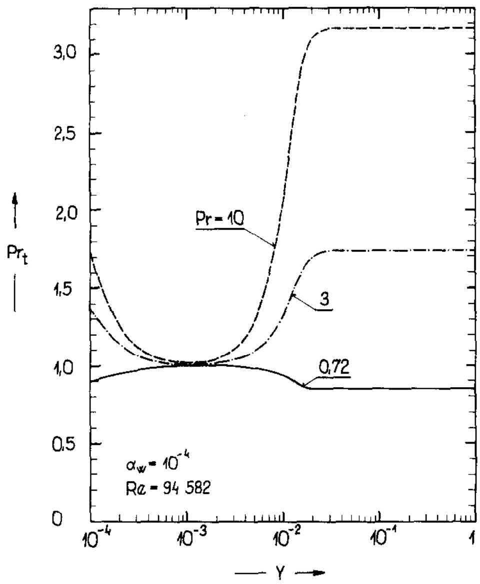

FIG. 21. Turbulent Prandtl number $Pr_t = f(Pr)$, $Re = 94\,582$.

which may, for the region near the wall, be simplified to

$$e^{-Y^2/\alpha}e^{-Y^2/\alpha_q}\left(\frac{1}{\alpha}-\frac{1}{\alpha_q}\right)-e^{-Y^2/\alpha_w}e^{-Y^2/\alpha_q}\left(\frac{1}{\alpha_w}-\frac{1}{\alpha_q}\right)$$
$$-e^{-Y^2/\alpha}e^{-Y^2/\alpha_{qw}}\left(\frac{1}{\alpha}-\frac{1}{\alpha_{qw}}\right)+e^{-Y^2/\alpha_w}e^{-Y^2/\alpha_{qw}}$$
$$\times\left(\frac{1}{\alpha_w}-\frac{1}{\alpha_{qw}}\right)=0, \quad (5.20a)$$

fulfilled identically on the wall ($Y = 0$). The relation (5.20a) is based on the simplified eddy viscosity and the eddy diffusivity of heat limited to two basic branches (with terms containing Y only). For the sake of the symmetry of the eddy viscosity and eddy diffusivity, according to the model, the extreme turbulent Prandtl number must be even at the center line ($Y = 1, R = 0$). The center line value of the turbulent Prandtl number is determined by

$$Pr_{t0} = \frac{A(e^{-1/\alpha}-e^{-1/\alpha_w})}{A_q(e^{-1/\alpha_q}-e^{-1/\alpha_{qw}})}. \quad (5.21)$$

For high Reynolds numbers (low α_{qw} and α_w) and $\alpha_q = \alpha$ this can be simplified for classical fluids to

$$Pr_{t0} \doteq Pr^{1/2}. \quad (5.21a)$$

As well as the extremes on the wall and at the center line, generally there are two other extremes of the local turbulent Prandtl number Pr_t in the interval $Y \in \langle 0, 1\rangle$.

For individual flow regimes over a selected range of Prandtl number values $Pr \in \langle 0.003, 0.1\rangle$ covering practically the total region of liquid metals, the values of the Nusselt number Nu have been fixed with the aid of equation (4.5a) and the temperature at the center line from equation (4.4). The computed values of the Nusselt number for the selected values of Pr as a function of the Reynolds number Re are given in Fig. 23. In Fig. 24 values of Nu are plotted as a function of Peclet numbers ($Pe = Re\,Pr$) and in Fig. 25 values of the temperature Θ_0 at the center line are plotted as a function of the Reynolds number Re. The dependences are limited by the line corresponding to zero contribution by the turbulent heat transport $q_t = 0$, i.e. the regime corresponding to $\alpha_{qw} = \alpha_q = 1/\pi$. The broken limiting line corresponds to the transition flow regime for $Re \gtrsim 7000$ for which the coefficients α ($< 1/\pi$), α_w ($< \alpha$) and A have been computed from the experimental data of the friction factor f and the velocities at the center line U_0 [16]. The agreement of the computed values of the Nusselt number Nu with the data found in the literature for the region of fully developed convection of liquid metals is apparent.

It seems necessary to point out that the model in question is valid for fully developed convective heat transport only, i.e. for high values of Peclet number Pe. For low values of $Pe \gtrsim 500$, the computed values of heat transfer are lower and the values of the temperature Θ_0 in the pipe axis are higher than in reality. This means that the results obtained when the model is used for low values of Pe are more reliable. This is due to the fact that for low Pe values $\alpha_q < \alpha$ and $\alpha_{qw} < \alpha_w/Pr$; as for velocity profiles in the area of low Reynolds numbers, $Re \gtrsim 7000$, it is necessary to consider $\alpha < 1/\pi$ when applying the eddy viscosity model [16].

6. RECOVERY FACTOR

The determination of heat transfer described in the preceding section may only be used when the influence of dissipated energy on the temperature profile need not be considered and when the dissipated energy is negligible. When the gases flow at high velocities the influence of the dissipated energy on heat transfer is considerable.

Knowledge of the eddy viscosity and eddy diffusivity

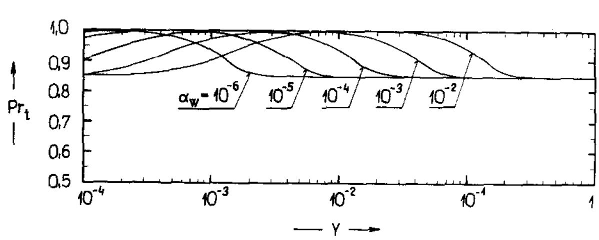

FIG. 22. Turbulent Prandtl number $Pr_t = f(\alpha_w)$, $Pr = 0.72$.

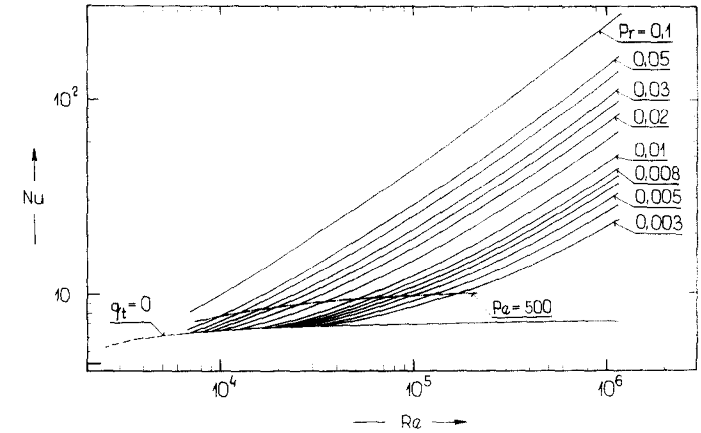

FIG. 23. Nusselt number Nu in dependence on Reynolds number Re for liquid metals ($Pr = 0.003$–0.1).

of heat allows the influence of the dissipated energy on heat transfer at high velocities to be determined with the aid of the recovery factor r^{**} and the adiabatic wall temperature $T_{w\,ad}$, and the determination of the dissipative heat transfer for the removal of dissipated energy. Although the heat transfer influenced by energy dissipation must be taken into account in compressible fluids, it is possible to base further considerations on the relations derived for incompressible fluids and take the results for limiting values.

The energy equation of turbulent flow (2.8), after being multiplied by ρ, gives the local density of the dissipated energy in unit time (W m^{-3}). Using the RHS of equation (2.8) it is possible to write for the local density of heat sources in the turbulent fluid flow

$$\tilde{q} = \rho \frac{r}{r_w} u^{*2} \frac{du}{dr} = \frac{r}{r_w} \rho \frac{f}{2} u_s^2 \frac{du}{dr}. \qquad (6.1)$$

To describe the dissipated energy transport across the flow, equation (4.1) without its RHS may be used. The zero value of the RHS corresponds to the density of the wall heat flux relevant to the dissipated energy; in this case any medium temperature increase does not occur along the path, i.e. $\partial T/\partial x = 0$. The source term $\tilde{q}$, however, must be added to the equation. The differential equation describing the thermal conditions in the medium on removal of the dissipated energy has the form

$$\frac{1}{r}\frac{d}{dr}\left[(\lambda + \rho c_p \varepsilon_q) r \frac{dT}{dr}\right] + \tilde{q} = 0, \qquad (6.2)$$

where $\tilde{q}$ is determined from equation (6.1). Using the equation for the heat flux density at the wall corresponding to the dissipated energy

$$q_w^{**} = \tau_w u_s = \frac{f}{2} \rho u_s^3 = h^{**} \Delta T^{**}, \qquad (6.3)$$

equation (6.2) may be transformed into the dimensionless form

$$\frac{1}{R}\frac{d}{dR}\left[\frac{a+\varepsilon_q}{a} R \frac{d\Theta^{**}}{dR}\right] = \frac{1}{2}\frac{f}{4} Re \frac{R^2}{(\nu+\varepsilon)/\nu} Nu^{**} \qquad (6.4)$$

where

$$\Theta^{**} = \frac{(T - T_w)^{**}}{\Delta T^{**}}$$

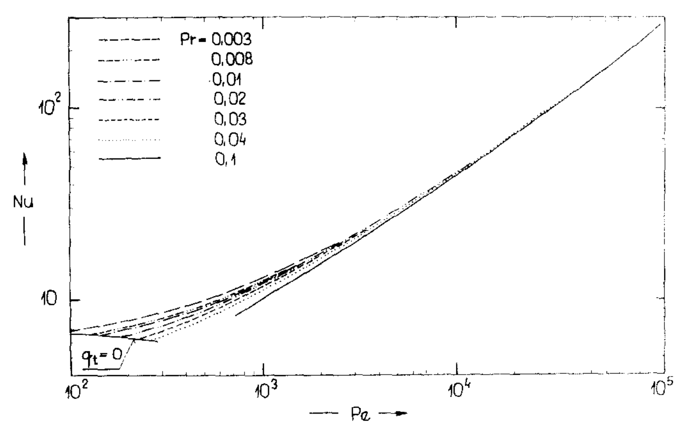

FIG. 24. Nusselt number Nu in dependence on Peclet number Pe for liquid metals ($Pr = 0.003$–0.1).

and

$$Nu^{**} = \frac{2h^{**}r_w}{\lambda}.$$

The term $R^2/[(\nu+\varepsilon)/\nu]$, according to the equation (2.8e), corresponds to the local total dissipated energy rate.

After the differential equation (6.4) has been integrated twice for the boundary conditions $R = 0 \Rightarrow \mathrm{d}\Theta^{**}/\mathrm{d}R = 0$ and $R = 1 \Rightarrow \Theta^{**} = 0$ the following equation for the temperature profile is obtained:

$$\Theta^{**} = \frac{1}{2}\frac{f}{4}Re\,Nu^{**} \times \int_R^1 \frac{\int_0^R \{R^3/[(\nu+\varepsilon)/\nu]\}\,\mathrm{d}R}{R[(a+\varepsilon_q)/a]}\,\mathrm{d}R. \quad (6.5)$$

After inserting equation (6.5) into the normalizing condition (4.3a), using equation (2.5) for the fluid velocity, and integrating the expression for the Nusselt number corresponding to the dissipated energy removal, the result is

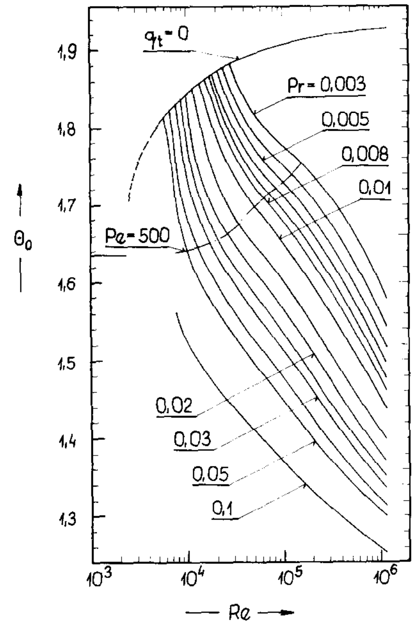

FIG. 25. Relative center line temperature Θ_0 for liquid metals.

$$Nu^{**} = \frac{1}{[(f/4)Re]^2\int_0^1 R\left(\int_R^1\{R/[(\nu+\varepsilon)/\nu]\}\mathrm{d}R\right)\int_R^1\frac{\int_0^R \frac{R^3}{[(\nu+\varepsilon)/\nu]}\mathrm{d}R}{R[(a+\varepsilon_q)/a]}\mathrm{d}R\,\mathrm{d}R}. \quad (6.6)$$

The temperature gradient at the wall then is

$$\left(\frac{\mathrm{d}\Theta^{**}}{\mathrm{d}R}\right)_w = -\frac{Nu^{**}}{2}. \quad (6.7)$$

From equation (6.3) the temperature difference between the medium and the wall corresponding to the dissipated energy removal may be determined as

$$\Delta T^{**} = \frac{q_w^{**}}{h^{**}} = r_q^{**}\frac{u_s^2}{2c_p}, \quad (6.8)$$

which may be expressed by means of temperature increase $u_s^2/2c_p$ corresponding to the total annihilation of kinetic energy of the flow, as well as the auxiliary complementary factor r_q^{**}. Equation (6.8) may be then adapted into the dimensionless form

$$r_q^{**} = \frac{f\,Re\,Pr}{Nu^{**}}. \quad (6.8a)$$

For the auxiliary complementary factor r_q^{**} the following expression results from the equations (6.6) and (6.8)

$$r_q^{**} = 4\left(\frac{f}{4}Re\right)^3 Pr\int_0^1 R\left(\int_R^1\frac{R}{(\nu+\varepsilon)/\nu}\mathrm{d}R\right)\int_R^1\frac{\int_0^R\frac{R^3}{(\nu+\varepsilon)/\nu}\mathrm{d}R}{R[(a+\varepsilon_q)/a]}\mathrm{d}R\,\mathrm{d}R. \quad (6.8b)$$

For the laminar flow where $\varepsilon/\nu = \varepsilon_q/a \equiv 0$ and $(f/4)Re \equiv 4$,

$$r_q^{**} = \tfrac{5}{3}Pr.$$

The chosen approach enables heat transfer at high flow velocities to be interpreted as the superposition of convective and dissipative heat transfer. This is made possible by the fact that the initial differential equations (4.1a) and (6.4) are linear. It is then possible to superpose the temperature profiles (expressed in dimensional form). Heat flux densities at the wall may be also superposed if the corresponding direction of heat flow is respected.

For several flow regimes (defined by the value α_w) of media with $Pr = 0.72$ the values of the auxiliary complementary factor r_q^{**} determined from equation (6.8b) are given in Table 3. In Table 4 the values of the factor r_q^{**} for media with $Pr = 0.67$, 0.72 and 0.8 for $\alpha_w = 0.00001$ ($Re = 334\,588$) are given. The values of Nusselt number Nu^{**} for dissipated energy transfer determined from equation (6.6), as well as the values of

Table 3. Recovery factor $r^{**} = f(Re)$, $Pr = 0.72$

α_w	lamin.	0.0001	0.00001	0.000002
Re	$\leqslant 2300$	94582.349	334588.29	803923.29
r_q^{**}	$5/3\ Pr = 1.2$	0.85546253	0.84291731	0.84261433
$Nu^{**} = f\ Re\ Pr/r_q^{**}$	$48/5 = 9.6$	362.4542	1012.9873	2077.8885
Nu	$48/11 = 4.\overline{36}$	183.1476	507.7732	1043.4736
$K^{**} = Nu^{**}/Nu$	$11/5 = 2.2$	1.9790	1.9950	1.9913
$r^{**} = (K^{**}-1)r_q^{**}$	$2\ Pr = 1.44$	0.837497817	0.838702724	0.835283585

convective Nusselt number Nu determined from the equation (4.5a) and the ratio $K^{**} = Nu^{**}/Nu$ (the ratio of dissipative and convective heat transfer) are also given in these tables.

Heat transfer at high flow velocities is defined by [28]

$$h^{***} = \frac{q_w}{T_{w\,ad} - T_w}, \tag{6.9}$$

where

$$T_{w\,ad} = T_s + \Delta T_{ad}^{**} = T_s + r^{**}\frac{u_s^2}{2c_p} \tag{6.10}$$

signifies the adiabatic wall temperature. The recovery factor r^{**} gives the quotient of the adiabatic wall temperature increase ΔT_{ad}^{**} due to flow energy dissipation and of the temperature increase $u_s^2/2c_p$ corresponding to the total annihilation of flow energy expressed by the bulk velocity u_s

$$r^{**} = \frac{\Delta T_{ad}^{**}}{(u_s^2/2c_p)}. \tag{6.10a}$$

The adiabatic wall temperature $T_{w\,ad}$ or the value of the recovery factor r^{**} may be determined by the superposition of the dissipative and convective heat transfer if the heat flux supplied from outside q_w is considered identical with the heat flux at the wall q_w^{**} at the dissipated energy removal. For this limiting case

$$\Delta T_{ad}^{**} = \Delta T - \Delta T^{**}. \tag{6.11}$$

It follows from the equality $q_w = (-)q_w^{**}$ that

$$h\Delta T = h^{**}\Delta T^{**} \tag{6.12}$$

and therefore it is possible to write

$$\frac{\Delta T}{\Delta T^{**}} = \frac{h^{**}}{h} = \frac{Nu^{**}}{Nu} = K^{**}. \tag{6.12a}$$

For the recovery factor, with regard to the equations (6.8) and (6.12a), we may write

$$r^{**} = (K^{**}-1)r_q^{**}. \tag{6.13}$$

The values of the recovery factor r^{**} determined from the equation (6.13) are given in Tables 3 and 4. The resultant values lie in close proximity to the value $Pr^{1/2}$ which is used in practical computations based on the experimental data as the value of the recovery factor r^{**} in the pipe [18, 21]. For laminar flow in the pipe $r^{**} = 2Pr$.

7. CONCLUSION

The models in question regard the eddy viscosity and eddy diffusivity of heat as the probability density distributions of the solid wall influence into the fluid flow. The character of the eddy viscosity and the eddy diffusivity of heat, described by reputable authors, and their proportionality near the wall to the third or fourth power of the distance from the wall, suggest that the eddy viscosity and the eddy diffusivity of heat result from two similar stochastic processes acting in opposite directions. The former process occurs mainly in the turbulent core and is practically identical for all regimes of fully developed flow. The latter process occurs mainly in the proximity of the wall and its reach decreases in flow regimes defined by high Reynolds numbers.

The model of eddy viscosity accounts for some facts found experimentally in turbulent flow. In the first place there is the physical explanation of the position of the boundary between the turbulent core and the buffer layer, usually at the point where $y^+ = 27$–30, by the position of the inflexion point of the eddy viscosity curve. The turbulent energy density maximum production position corresponds to the values of the boundary between the turbulent core and the viscous layer given by the Blasius relation. The value $y^+ = 5$ is usually regarded as the viscous sublayer in which the contribution of turbulent transport may be neglected. According to the model there is a ratio $\varepsilon/\nu \doteq 0.1$ corresponding to this boundary. For the ratio ε/ν

Table 4. Recovery factor $r^{**} = f(Pr)$, $Re = 334\ 588$

Pr	0.67	0.72	0.8
r_q^{**}	0.79833659	0.84291731	0.91266830
$Nu^{**} = f\ Re\ Pr/r_q^{**}$	995.27999	1012.98730	1039.52162
Nu	494.10933	507.77320	528.9445
$K^{**} = Nu^{**}/Nu$	2.0142910	1.99496015	1.96527541
$r^{**} = (K^{**}-1)r_q^{**}$	0.8097456	0.83870272	0.88097627
$Pr^{1/2}$	0.81853528	0.84852814	0.89442719

$= 0.01$ there is a corresponding dimensionless distance from the wall $y^+ \doteq 2.5$.

The advantage of these models is in the fact that their mathematical expression makes use of normalized functions so that the proportionality coefficients represent the mean values of the quantities in question. The coefficients A_t or A_{qt} represent directly the mean value of the relative eddy viscosity $(\varepsilon/\nu)_s$ or eddy diffusivity of heat $(\varepsilon_q/a)_s$, and are, moreover, in their dimensionless expression $A_t^+ = \varepsilon_s^+$ or $A_{qt}^+ = \varepsilon_{qs}^+$ practically constant over the whole area of the developed flow.

The connections between the models of eddy viscosity and eddy diffusivity of heat are expressed by means of a simple bond between the coefficients which occur in the analytical expressions describing the models, namely by the factor $Pr^{+1/2}$ or $Pr^{-1/2}$ in classical fluids and by the factor Pr^{+1} or Pr^{-1} in liquid metals. These bonds correct the analogy used by some authors between heat transfer and momentum transfer which is based on the direct proportionality between the eddy viscosity and the eddy diffusivity of heat.

The agreement of the basic flow and thermokinetic characteristics determined according to the models with the experimental data about these quantities stated by both reputable authors and the basic literature on this subject is so obvious that any comparison seems needless.

The model of eddy diffusivity of heat may also be used for the description of mass transfer conditions. All the relations given above may be used after substituting Schmidt number Sc for Prandtl number Pr. In a general sense though, there are more similarities between mass transfer and heat transfer at a constant wall temperature, $T_w = \text{const}$. The influence of the wall temperature distribution, or possibly of the wall heat flux distribution, on heat transfer plays an important role especially in fluids with very low Prandtl numbers, i.e. in liquid metals.

The dimensionless differential equation of heat transfer for the case of boundary condition $T_w = \text{const.}$ has the form [22]

$$\frac{1}{R}\frac{d}{dR}\left[\frac{a+\varepsilon_q}{a}R\frac{d\Theta_T}{dR}\right] = -\Theta_T U\, Nu_T. \qquad (7.1)$$

From its solution the relation for the temperature profile follows as

$$\Theta_T = Nu_T \int_R^1 \frac{\int_0^R \Theta_T UR\, dR}{R[(a+\varepsilon_q)/a]}\, dR \qquad (7.2)$$

as well as the relation for the Nusselt number [22]

$$Nu_T = \frac{1}{2\int_0^1 UR \int_R^1 \left\{\frac{\int_0^R \Theta_T UR\, dR}{R[(a+\varepsilon_q)/a]}\right\} dR\, dR}. \qquad (7.3)$$

Formally identical dependences which differ only in the terms τ/τ_w and q/q_w are valid both for heat and momentum transport. In the case of flow, the decisive quantity is the shear stress distribution τ/τ_w. In the case of heat transport, heat flux density distribution, q/q_w, is decisive, being dependent on the thermal boundary conditions. With the boundary condition $q_w = \text{const.}$, the ratio q/q_w depends exclusively on the flow conditions.

The eddy viscosity model enables the local direct viscous dissipated energy rate W_v to be determined as well as the turbulent energy production rate W_t also transformed by dissipation into heat. Knowledge of the local heat sources in the flow, in connection with the eddy diffusivity of heat model, enables the influence of medium flow velocity on heat transfer (so called recovery factor r^{**}) to be determined, as well as the dissipative heat transfer h^{**}.

Both the eddy viscosity and eddy diffusivity of heat models may also be applied to other geometrical configurations of channels and, after some adjustments, to flow along solid surfaces.

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APPENDIX

DISCUSSION OF EDDY VISCOSITY AND EDDY DIFFUSIVITY OF HEAT MODELS

Connection of the models with mixing length and with mixing velocity

Eddy viscosity model. The suggested model of eddy viscosity is in full agreement with the conception of turbulent momentum transport produced by Prandtl and perfected by Kármán, which is based on an analogy with the mechanism of molecular viscosity from the kinetic theory of gas. According to this theory the molecular kinematic viscosity is proportional to the product of the mean translational velocity of molecules $\bar{v}$ and the mean free path ℓ [32]

$$\nu = 0.499\bar{v}\ell. \tag{A1}$$

Analogically, eddy viscosity may also be expressed as proportional to the product of the transverse mixing velocity $\mathfrak{v}$ ($[\mathfrak{v}] = \mathrm{m\,s^{-1}}$) and the turbulent mixing length l_t, ($[l_t] = \mathrm{m}$), as in the relation

$$\varepsilon = 2\mathfrak{v}l_t, \tag{A2}$$

which may be rewritten into the dimensionless form

$$\frac{\varepsilon}{\nu} = \frac{2\mathfrak{v}r_w}{\nu}\frac{l_t}{r_w} = Re_\varepsilon L_t. \tag{A3}$$

When compared with equation (3.20a) for the eddy viscosity, the meaning of the terms and coefficients in this expression will be obvious. The expression in square brackets [equation (3.20a)] corresponds to the resultant turbulent relative mixing length L_t and the expression $2A/\alpha$ before the brackets to the dimensionless transverse mixing velocity

$$\frac{2A}{\alpha} = \frac{2\mathfrak{v}r_w}{\nu} = Re_\varepsilon. \tag{A4}$$

The expression in brackets may be expressed according to the equation (3.20) as the difference between two partial expressions containing only the terms with the coefficient α or α_w, each of them corresponding to the relative length (the relative mixing length in particular),

$$L = Y e^{-Y^2/\alpha} + (2-Y)e^{-(2-Y)^2/\alpha} - (2+Y)e^{-(2+Y)^2/\alpha} - (4-Y)e^{-(4-Y)^2/\alpha}, \tag{A5}$$

which dominates in the turbulent core and of the relative length

$$L_w = Y e^{-Y^2/\alpha_w} + (2-Y)e^{-(2-Y)^2/\alpha_w} - (2+Y)e^{-(2+Y)^2/\alpha_w}(4-Y)e^{-(4-Y)^2/\alpha_w} \tag{A6}$$

limiting the first length in the region near the wall,

$$L_t = L - L_w = H(Y). \tag{A7}$$

The relative mixing velocity $\mathscr{V} = \mathfrak{v}/u_s$ may be obtained from the ratio of the dimensionless mixing velocity Re_ε and Reynolds number Re corresponding to the bulk velocity

$$\mathscr{V} = \frac{Re_\varepsilon}{Re}. \tag{A8}$$

The relative transverse mixing velocity $\mathscr{V}$ is shown as a function of α_w in Fig. 26 together with the values of the velocity $\mathscr{V}$ determined on the basis of the model with the use of the coefficients obtained from Nikuradse's and Rothfus's data.

The course of the relative turbulent mixing length $L_t = l_t/r_w$ along the pipe radius for $\alpha = 1/\pi$ and for a few values of the

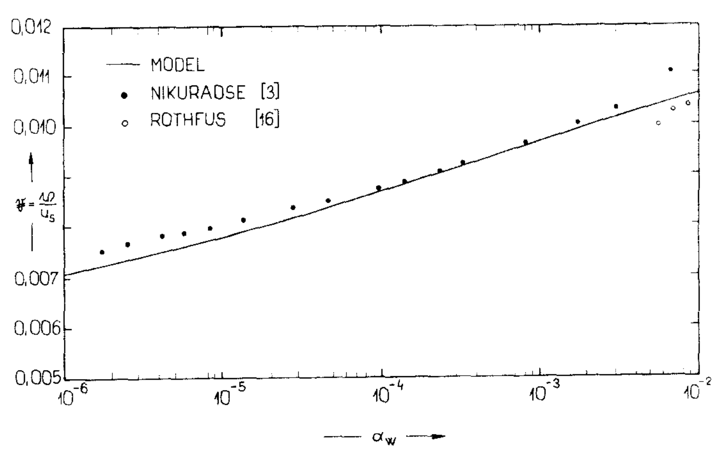

FIG. 26. Relative mixing velocity.

coefficient α_w is shown in Fig. 27. For the sake of comparison the mixing length according to Nikuradse expressed by the equation (A9) [15] is also shown,

$$L = 0.14 - 0.08R^2 - 0.06R^4. \tag{A9}$$

The value of the integrals expressing the mean value of the relative mixing lengths is equal to one half of the coefficient values α and α_w

$$L_s = \frac{l_s}{r_w} = \int_0^1 L \, dY = \frac{\alpha}{2}, \tag{A10a}$$

$$L_{ws} = \frac{l_{ws}}{r_w} = \int_0^1 L_w \, dY = \frac{\alpha_w}{2}, \tag{A10b}$$

$$L_{ts} = L_s - L_{ws} = \frac{\alpha - \alpha_w}{2}. \tag{A10c}$$

For the coefficients A, A_w, A_t there are the following expressions:

$$A = \frac{2\mathfrak{v} l_s}{\nu} = Re_\varepsilon L_s, \tag{A11a}$$

$$A_w = \frac{2\mathfrak{v} l_{ws}}{\nu} = Re_\varepsilon L_{ws}, \tag{A11b}$$

$$A_t = \frac{2\mathfrak{v} l_{ts}}{\nu} = Re_\varepsilon L_{ts}. \tag{A11c}$$

If we know the transverse mixing velocities $\mathfrak{v}$ or $\mathscr{V}$ or Re_ε and the mean mixing lengths l_s or L_s or L_{ws} and L_{ts} the time scale of existence of the turbulent elements in the core, as well as in the flow region near the wall, can be determined

$$t_s = \frac{l_s}{\mathfrak{v}}, \tag{A12a}$$

$$t_{ws} = \frac{l_{ws}}{\mathfrak{v}}, \tag{A12b}$$

$$t_{ts} = t_s - t_{ws} = \frac{l_{ts}}{\mathfrak{v}}. \tag{A12c}$$

In the dimensionless expression by Zhukowsky number Zh [20] the time characteristics will have the form

$$Zh = \frac{\nu t_s}{r_w^2} = \frac{2L_s}{Re_\varepsilon} = \frac{\alpha\alpha_w}{2A_w}, \tag{A12d}$$

$$Zh_w = \frac{\nu t_{ws}}{r_w^2} = \frac{2L_{ws}}{Re_\varepsilon} = \frac{\alpha\alpha_w}{2A}, \tag{A12e}$$

$$Zh_t = \frac{\nu t_{ts}}{r_w^2} = \frac{2L_{ts}}{Re_\varepsilon} = Zh - Zh_w = \frac{\alpha\alpha_w}{2}\left(\frac{1}{A_w} - \frac{1}{A}\right). \tag{A12f}$$

The mean values of the relative mixing lengths L_s, L_{ws}, L_{ts} and the respective values of the Zhukowsky number Zh, Zh_w, Zh_t are given for several values of the coefficient α_w in Table 5, together with the values of the respective relative mixing velocity $\mathscr{V}$ or Re_ε.

Equation (3.22a) for the coefficient K in equation (3.21) may be rewritten using the Zhukowsky number Zh corresponding to the turbulent core and Zh_w for the region near the wall into the form

$$K = \frac{2A_t}{\alpha\alpha_w} = \frac{2A}{\alpha\alpha_w}\left(1 - \frac{\alpha_w}{\alpha}\right) = \frac{1}{Zh_w} - \frac{1}{Zh}. \tag{A13}$$

Eddy diffusivity of heat model. The explanation of the physical meaning of the coefficients A_q, α_q and α_{qw} occurring in equation (5.3) for eddy diffusivity of heat may be based, similarly as in the case of eddy viscosity, on an analogy between molecular and turbulent transport. For molecular thermal diffusivity, from the kinetic gas theory the following

Table 5. Mixing length and mixing velocity

	α_w					
	10^{-6}	10^{-5}	10^{-4}	10^{-3}	10^{-2}	
$Re_\varepsilon = 2A/\alpha = 2\mathfrak{v}\tau_w/\nu$	8280.32	2613.88	822.53	256.56	79.06	
$\mathscr{V} = \mathfrak{v}/u_s = Re_\varepsilon/Re$	0.007084	0.007812	0.008696	0.009701	0.010653	
$L_s = \alpha/2$			0.1591549431			
$L_{ws} = \alpha_w/2$	0.0000005	0.000005	0.00005	0.0005	0.005	
$L_{ts} = (\alpha - \alpha_w)/2$	0.159154443	0.159149943	0.159104943	0.158654943	0.154154943	
$Zh = \alpha/Re_\varepsilon = t_s\nu/r_w^2$	$3.844171274 \times 10^{-5}$	$1.217769726 \times 10^{-4}$	$3.869873828 \times 10^{-4}$	$1.24066322 \times 10^{-3}$	$4.025907901 \times 10^{-3}$	
$Zh_w = \alpha_w/Re_\varepsilon = t_{ws}\nu/r_w^2$	$1.2076820 \times 10^{-10}$	3.8257364×10^{-9}	1.2157567×10^{-7}	3.8976584×10^{-6}	1.2647762×10^{-4}	
$Zh_t = (\alpha - \alpha_w)/Re_\varepsilon = t_{ts}\nu/r_w^2$	3.8441591×10^{-5}	1.2177314×10^{-4}	3.868658×10^{-4}	1.2367655×10^{-3}	3.8994302×10^{-3}	
$(\tau_t/\tau_w)_{max}$	0.9781	0.9615	0.9321	0.8805	0.7727	
f	0.002791	0.003523	0.004548	0.006101	0.008499	Model
Re	1187784	336625	94690	26237	7313	Model
f	0.002836	0.003544	0.004553	0.006053	0.0083737	Nikuradse
Re	1168946	334588	94582	26447	7422	Nikuradse

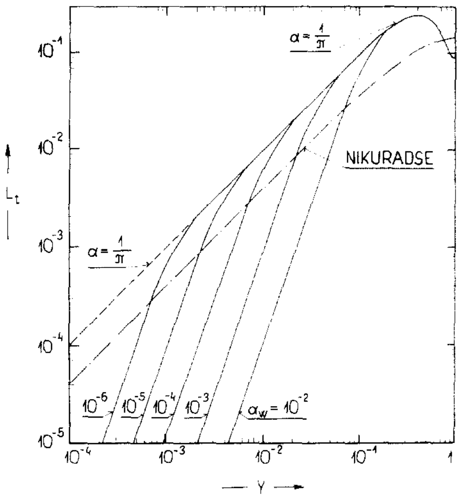

FIG. 27. Relative mixing length.

equation is obtained [32]

$$a = 1.259 \frac{\bar{v}\ell}{(c_p/c_v)} \quad \text{(A14)}$$

stating its proportionality, as in the case of molecular viscosity, to the product of the mean free path ℓ and the mean translational velocity of molecules $\bar{v}$. With the use of the ratio $\nu/a = Pr$ and equation (A1) for molecular viscosity we may write

$$a = 0.499 \frac{\bar{v}\ell}{Pr}. \quad \text{(A15)}$$

The eddy diffusivity of heat, by analogy with the eddy viscosity, may be put proportional to the transverse mixing velocity $\mathfrak{v}$ and to the difference between the two partial characteristic lengths L and L_w. Each of the terms, however, will be corrected by the complementary function of the Prandtl number Pr or possibly of the distance from the wall Y

$$\frac{\varepsilon_q}{a} = \frac{2\mathfrak{v}r_w}{\nu}\frac{\nu}{a} f(Pr)[L\,g(Pr,Y) - L_w\,g_w(Pr,Y)]. \quad \text{(A16)}$$

From a comparison with equation (5.3) for the relative eddy diffusivity of heat, the complementary functions $f(Pr)$, $g(Pr, Y)$, and $g_w(Pr, Y)$ may be determined. For $\alpha_q = \alpha$ the complementary function is

$$g(Pr, Y) = 1. \quad \text{(A17)}$$

For the function $g_w(Pr, Y)$ at $\alpha_{qw} = \alpha_w/(Pr^{1/2})$ valid for classical fluids it is possible to write

$$g_w(Pr,Y) = \frac{Y\,e^{-Y^2Pr^{1/2}/\alpha_w} + (2-Y)e^{-(2-Y)^2Pr^{1/2}/\alpha_w} - (2+Y)e^{-(2+Y)^2Pr^{1/2}/\alpha_w} - (4-Y)e^{-(4-Y)^2Pr^{1/2}/\alpha_w}}{Y\,e^{-Y^2/\alpha_w} + (2-Y)e^{-(2-Y)^2/\alpha_w} - (2+Y)e^{-(2+Y)^2/\alpha_w} - (4-Y)e^{-(4-Y)^2/\alpha_w}}. \quad \text{(A18)}$$

As the terms including $(2-Y)$, $(2+Y)$ and $(4-Y)$ have very low values, it is possible to simplify this to

$$g_w(Pr,Y) \doteq \frac{e^{-Y^2Pr^{1/2}/\alpha_w}}{e^{-Y^2/\alpha_w}} = (e^{-Y^2/\alpha_w})^{(Pr^{1/2}-1)}. \quad \text{(A18a)}$$

For the complementary function $f(Pr)$ at $A_q = A\,Pr^{1/2}$, valid for classical fluids,

$$f(Pr) = \frac{1}{Pr^{1/2}}. \quad \text{(A19)}$$

The relative eddy diffusivity of heat is then connected with the dimensionless mixing velocity Re_ε or Pe_ε and the local values of the characteristic mixing lengths of the turbulent flow L and L_w by

$$\frac{\varepsilon_q}{a} = Re_\varepsilon Pr^{1/2}[L - L_w(e^{-Y^2/\alpha_w})^{(Pr^{1/2}-1)}]$$
$$= \frac{Pe_\varepsilon}{Pr^{1/2}}[L - L_w(e^{-Y^2/\alpha_w})^{(Pr^{1/2}-1)}]. \quad \text{(A20)}$$

For the relative mean eddy diffusivity of heat, the dependence on the values of the characteristic mixing lengths L_s and L_{ws} is

$$\left(\frac{\varepsilon_q}{a}\right)_s = A_{qt} = Re_\varepsilon[L_s\,Pr^{1/2} - L_{ws}] = \frac{Pe_\varepsilon}{Pr^{1/2}}\left[L_s - \frac{L_{ws}}{Pr^{1/2}}\right]. \quad \text{(A21)}$$

For liquid metals, for which the coefficients of eddy viscosity and eddy diffusivity of heat models in the region of developed convection are connected by the relations $A_q = A\,Pr$ and $\alpha_{qw} = \alpha_w/Pr$, for $\alpha_q = \alpha$ the complementary functions result in

$$f(Pr) = 1, \quad \text{(A22)}$$

$$g(Pr, Y) = 1, \quad \text{(A17)}$$

$$g_w(Pr, Y) = (e^{-Y^2/\alpha_w})^{(Pr-1)}. \quad \text{(A23)}$$

For the relative eddy diffusivity of heat of liquid metals

$$\frac{\varepsilon_q}{a} = Re_\varepsilon Pr[L - L_w(e^{-Y^2/\alpha_w})^{(Pr-1)}] \quad \text{(A24)}$$

with a mean value of

$$\left(\frac{\varepsilon_q}{a}\right)_s = A_{qt} = Re_\varepsilon[L_s\,Pr - L_{ws}] = Pe_\varepsilon\left[L_s - \frac{L_{ws}}{Pr}\right]. \quad \text{(A25)}$$

In view of the equations (A11a) and (A11b) it is possible to express the dependence of the coefficients A_q and A_{qw} on the transverse mixing velocity $\mathfrak{v}$ and the mixing lengths l_s and l_{ws}

$$A_{qw} = \frac{2\mathfrak{v}l_{ws}}{\nu}. \quad \text{(A26)}$$

For classical fluids

$$A_q = \frac{2\mathfrak{v}l_s}{(\nu a)^{1/2}} = \frac{2\mathfrak{v}l_s}{\nu}(Pr^{1/2}). \quad \text{(A27)}$$

and for liquid metals

$$A_q = \frac{2\mathfrak{v}l_s}{a} = \frac{2\mathfrak{v}l_s}{\nu}Pr. \quad \text{(A28)}$$

For the mean eddy diffusivity of heat of classical fluids the following expression may be derived:

$$\varepsilon_{qs} = 2\mathfrak{v}\left[\frac{l_s}{Pr^{1/2}} - \frac{l_{ws}}{Pr}\right], \quad \text{(A29)}$$

which for liquid metals has the form

$$\varepsilon_{qs} = 2\mathfrak{v}\left[l_s - \frac{l_{ws}}{Pr}\right]. \quad \text{(A30)}$$

For a given regime of the fully developed flow and heat transport, the following relation between the coefficients of eddy viscosity and eddy diffusivity of heat is valid, independent of Prandtl number,

$$A\alpha_w = A_w\alpha = A_q\alpha_{qw} = A_{qw}\alpha_q = \text{const.} \tag{A31}$$

With the use of the equation (A4) for the dimensionless mixing velocity Re_ε and the relations (A10a) and (A10b) for the mean values of the relative mixing lengths L_s and L_{ws}, equation (A31) may be rewritten into a physically more illustrative form

$$A\alpha_w = 2Re_\varepsilon L_s L_{ws} = \text{const.} \tag{A31a}$$

in which thermophysical quantities do not occur.

Mathematical description of isolated vortex diffusion in connection with the eddy viscosity and eddy diffusivity of heat models

For the local vorticity of a diffusing isolated vortex [30, 31]

$$\Omega = \frac{\Gamma}{4\pi\nu t_\Gamma} e^{-r^{*2}/(4\nu t_\Gamma)}, \tag{A32}$$

while the initial circulation Γ may be expressed by

$$\Gamma = 2\pi r^* c_\Gamma = 2\pi r_w c_{\Gamma 0}, \tag{A33}$$

where c_Γ or $c_{\Gamma 0}$ is the circulating velocity on the general radius r^* or on the radius $r^* = r_w$. If equation (A32) is transformed into the dimensionless form

$$\frac{r_w r^*}{\nu}\Omega = \frac{2r_w c_{\Gamma 0}}{\nu}\frac{r^*}{r_w}\frac{1}{4\nu t_\Gamma/r_w^2} e^{-(r^{*2}/r_w^2)/(4\nu t_\Gamma/r_w^2)}. \tag{A34}$$

If the axis of the isolated vortex is located on the pipe wall we may write ($r^* = y$, $r^*/r_w = y/r_w = Y$)

$$\frac{r_w y}{\nu}\Omega = \frac{2r_w c_{\Gamma 0}}{\nu}\frac{1}{4\nu t_\Gamma/r_w^2} Y e^{-[Y^2/(4\nu t_\Gamma/r_w^2)]}. \tag{A35}$$

After insertion

$$\frac{2r_w c_{\Gamma 0}}{\nu} = Re_\Gamma = \frac{\Gamma}{\pi\nu} \tag{A36}$$

and

$$\frac{\nu t_\Gamma}{r_w^2} = \text{Zh}_\Gamma \tag{A37}$$

we will get

$$\frac{r_w y}{\nu}\Omega = \frac{Re_\Gamma}{4Zh_\Gamma} Y e^{-Y^2/4Zh_\Gamma}. \tag{A38}$$

The velocity fields of the two superposed diffusing isolated vortices with the initial circulation Γ and Γ_w in the opposite direction stopped at times t_Γ and $t_{\Gamma w}$ are described by

$$\frac{r_w y}{\nu}(\Omega - \Omega_w) = \frac{Re_\Gamma}{4Zh_\Gamma} Y e^{-Y^2/4Zh_\Gamma} - \frac{Re_{\Gamma w}}{4Zh_{\Gamma w}} Y e^{-Y^2/4Zh_{\Gamma w}}. \tag{A39}$$

From a comparison of equation (A39) with the basic branch of the eddy viscosity model [according to equation (3.19)]

$$\frac{\varepsilon}{\nu} \sim \frac{2A}{\alpha} Y e^{-Y^2/\alpha} - \frac{2A_w}{\alpha_w} Y e^{-Y^2/\alpha_w}$$

it follows for the relative eddy viscosity the proportion

$$\frac{\varepsilon}{\nu} \sim \frac{r_w y}{\nu}(\Omega - \Omega_w) \tag{A39a}$$

and for the individual coefficients the following expressions

$$\alpha = 4Zh_\Gamma\left(= \frac{1}{\pi}\right), \tag{A40a}$$

$$\alpha_w = 4Zh_{\Gamma w}, \tag{A40b}$$

$$A = \frac{Re_\Gamma}{2} = \frac{\Gamma}{2\pi\nu}, \tag{A41a}$$

$$A_w = \frac{Re_{\Gamma w}}{2} = \frac{\Gamma_w}{2\pi\nu}. \tag{A41b}$$

For the relative circulation velocity φ_Γ or $\varphi_{\Gamma w}$ it follows

$$\varphi_\Gamma = \frac{c_{\Gamma 0}}{u_s} = \frac{Re_\Gamma}{Re}, \tag{A42a}$$

$$\varphi_{\Gamma w} = \frac{c_{\Gamma 0 w}}{u_s} = \frac{Re_{\Gamma w}}{Re}. \tag{A42b}$$

The connection between the magnitude of the relative mixing velocity $\mathcal{V} = Re_\varepsilon/Re$, taken into account in the explanation of the eddy viscosity mechanism based on the mixing length, and the value of the relative circulation velocity φ_Γ or $\varphi_{\Gamma w}$ is expressed by the relations

$$\varphi_\Gamma = 4\mathcal{V} Zh_\Gamma = 2\mathcal{V} L_s, \tag{A42c}$$

$$\varphi_{\Gamma w} = 4\mathcal{V} Zh_{\Gamma w} = 2\mathcal{V} L_{ws}. \tag{A42d}$$

For $\alpha = 1/\pi$ we may write

$$\mathcal{V} = \pi\varphi_\Gamma.$$

With regard to equation (3.23) we may write

$$\frac{Re_\Gamma}{4Zh_\Gamma} = \frac{Re_{\Gamma w}}{4Zh_{\Gamma w}} = \frac{\Gamma}{4\pi t_\Gamma}\frac{r_w^2}{\nu^2} = \frac{\Gamma_w}{4\pi t_{\Gamma w}}\frac{r_w^2}{\nu^2} = \frac{c_{\Gamma 0}}{2t_\Gamma}\frac{r_w^3}{\nu^2} = \frac{c_{\Gamma 0 w}}{2t_{\Gamma w}}\frac{r_w^3}{\nu^2}. \tag{A43a}$$

For $\alpha = 1/\pi$ we may write

$$\frac{Re_\Gamma}{4Zh_\Gamma} = \frac{\Gamma}{\nu}. \tag{A43b}$$

For the relative eddy viscosity we may write

$$\frac{\varepsilon}{\nu} = \frac{Re_\Gamma}{4Zh_\Gamma}\left[Y(e^{-Y^2/4Zh_\Gamma} - e^{-Y^2/4Zh_{\Gamma w}}) + (2-Y)(e^{-(2-Y)^2/4Zh_\Gamma} - e^{-(2-Y)^2/4Zh_{\Gamma w}}) - (2+Y)(e^{-(2+Y)^2/4Zh_\Gamma} - e^{-(2+Y)^2/4Zh_{\Gamma w}}) - (4-Y)(e^{-(4-Y)^2/4Zh_\Gamma} - e^{-(4-Y)^2/4Zh_{\Gamma w}})\right] \tag{A44}$$

which corresponds to the course at different times t_Γ and $t_{\Gamma w}$ of the 'frozen' local vorticity of the two diffusing co-axial isolated vortices with their axis on the wall and with a different and opposite initial vorticity expressed by velocity circulation Γ and Γ_w, while the ratio of the initial circulations is identical with the ratio of the times t_Γ and $t_{\Gamma w}$

$$\frac{\Gamma}{\Gamma_w} = \frac{t_\Gamma}{t_{\Gamma w}}. \tag{A45a}$$

For $\alpha = 1/\pi$ we may write for time $t_\Gamma = r_w^2/(4\pi\nu)$ or

$$Zh_\Gamma = \frac{\alpha}{4} = \frac{1}{4\pi} = 0.0795774151. \tag{A45b}$$

Between the time scales used in considering the eddy viscosity mechanism on the basis of the mixing length, namely t_s and t_{ws} and the times t_Γ and $t_{\Gamma w}$ there is the proportionality

$$\frac{t_\Gamma}{t_s} = \frac{t_{\Gamma w}}{t_{ws}} = \frac{Zh_\Gamma}{Zh} = \frac{Zh_{\Gamma w}}{Zh_w} = \frac{A}{2\alpha} = \frac{A_w}{2\alpha_w} = \frac{Re_\varepsilon}{4}. \tag{A46a}$$

Consequently, there is also proportionality between the times

t_s and t_{ws} and the initial circulation

$$\frac{\Gamma}{\Gamma_w} = \frac{t_s}{t_{ws}}. \quad \text{(A46b)}$$

For $\alpha = 1/\pi$ we can write

$$Re_\varepsilon = \pi \, Re_\Gamma, \quad \text{(A47a)}$$

$$t_\Gamma = \frac{\pi \, Re_\Gamma}{4} t_s. \quad \text{(A47b)}$$

The proportionality coefficient K in the equation (3.21) may be expressed with the aid of Zhukowsky numbers Zh_Γ and $Zh_{\Gamma w}$

$$K = \frac{Re_\Gamma}{16 Zh_\Gamma}\left(\frac{1}{Zh_{\Gamma w}} - \frac{1}{Zh_\Gamma}\right) = \frac{Re_\Gamma}{16 Zh_\Gamma Zh_{\Gamma w}}\left(1 - \frac{Zh_{\Gamma w}}{Zh_\Gamma}\right). \quad \text{(A48)}$$

For the mean value of the relative eddy viscosity we may write

$$\left(\frac{\varepsilon}{\nu}\right)_s = \frac{Re_\Gamma - Re_{\Gamma w}}{2}. \quad \text{(A49)}$$

In the diffusion of the isolated vortex the change of the momentary circulation velocity $c_{\Gamma-t}$ to the stationary initial velocity c_Γ ratio is governed by the expression identical with Rayleigh distribution

$$\frac{d(c_{\Gamma-t}/c_\Gamma)}{dr^*} = \frac{2r^*}{4\nu t_\Gamma} e^{-r^{*2}/4\nu t_\Gamma}. \quad \text{(A50a)}$$

In the dimensionless expression the relation (A50a) has the form of

$$\frac{d(c_{\Gamma-t}/c_\Gamma)}{dY} = \frac{2Y}{4Zh_\Gamma} e^{-Y^2/4Zh_\Gamma}, \quad \text{(A50b)}$$

which follows from the derivation of the momentary velocity course (A51) that is formally identical with equation (3.10) for the distribution function $F(Y)$ [23, 24]

$$\frac{c_{\Gamma-t}}{c_\Gamma} = 1 - e^{-r^{*2}/4\nu t_\Gamma} = 1 - e^{-Y^2/4Zh_\Gamma} = F(Y). \quad \text{(A51)}$$

For eddy viscosity we can then write

$$\frac{\varepsilon}{\nu} \sim \frac{\Gamma}{2\pi\nu}\left(\frac{d(c_{\Gamma-t}/c_\Gamma)}{dY}\right)_{t=t_\Gamma} - \frac{\Gamma_w}{2\pi\nu}\left(\frac{d(c_{\Gamma-tw}/c_{\Gamma w})}{dY}\right)_{t=t_{\Gamma w}}. \quad \text{(A52)}$$

The connection between the momentary shear stress $\tau_{\Gamma-t}$ of the isolated vortex and the initial stationary shear stress τ_Γ is expressed directly by the change

$$\frac{d(c_{\Gamma-t}/c_\Gamma)}{dY} = \frac{r_w}{\nu}\left(\frac{\tau_{\Gamma-t}}{\rho}\frac{1}{c_\Gamma} - \frac{\tau_\Gamma}{\rho}\frac{c_{\Gamma-t}}{c_\Gamma^2}\right), \quad \text{(A53)}$$

where

$$\tau_{\Gamma-t} = \rho\nu \frac{dc_{\Gamma-t}}{dr^*}, \quad \text{(A54a)}$$

$$\tau_\Gamma = \rho\nu \frac{dc_\Gamma}{dr^*}. \quad \text{(A54b)}$$

A further expression of the eddy viscosity follows from a transformation of equation (A50b). If the momentary local vorticity Ω or Ω_w is replaced by the angle velocity ω or ω_w [30]

$$\omega = \frac{\Omega}{2}, \quad \text{(A55a)}$$

$$\omega_w = \frac{\Omega_w}{2} \quad \text{(A55b)}$$

and if the momentary fictitious peripheral velocity of vortices is introduced

$$c_{\omega y} = \omega y \quad \text{(A56a)}$$

or

$$c_{\omega yw} = \omega_w y \quad \text{(A56b)}$$

we may obtain an expression equivalent to the first term of equation (3.20a) for the eddy viscosity

$$\frac{\varepsilon}{\nu} \sim \frac{2r_w c_{\omega y}}{\nu} - \frac{2r_w c_{\omega yw}}{\nu}. \quad \text{(A57)}$$

If we put

$$\frac{2r_w c_{\omega y}}{\nu} = Re_{\omega y} \quad \text{(A58a)}$$

or

$$\frac{2r_w c_{\omega yw}}{\nu} = Re_{\omega yw}, \quad \text{(A58b)}$$

we can then write

$$\frac{\varepsilon}{\nu} \sim Re_{\omega y} - Re_{\omega yw}. \quad \text{(A59)}$$

The fictitious peripheral vortex velocity may also be expressed as a relative one

$$\varphi_{\omega Y} = \frac{c_{\omega y}}{u_s} = \frac{Re_{\omega y}}{Re} \quad \text{(A60a)}$$

or

$$\varphi_{\omega Yw} = \frac{c_{\omega yw}}{u_s} = \frac{Re_{\omega yw}}{Re}. \quad \text{(A60b)}$$

Relations analogous to equations (A56), (A59) and (A60) may also be written for other terms of the eddy viscosity expression containing the coordinates $(2 - Y)$ or $(2 + Y)$ and $(4 - Y)$. The momentary fictitious peripheral velocities are connected with the components of the relative mixing length by

$$\frac{\varphi_{\omega Y}}{\mathscr{V}} = \frac{c_{\omega y}}{\mathfrak{v}} = \frac{l_y}{r_w} = L_Y \quad \text{(A60c)}$$

or

$$\frac{\varphi_{\omega Yw}}{\mathscr{V}} = \frac{c_{\omega yw}}{\mathfrak{v}} = \frac{l_{wy}}{r_w} = L_{wY}. \quad \text{(A60d)}$$

The mean value of the relative eddy viscosity may then be expressed by means of the mean momentary fictitious peripheral velocity $c_{\omega s}$ or $c_{\omega ws}$

$$\left(\frac{\varepsilon}{\nu}\right)_s = \frac{2r_w c_{\omega s}}{\nu} - \frac{2r_w c_{\omega ws}}{\nu} = Re_{\omega s} - Re_{\omega ws} \quad \text{(A61)}$$

where

$$\frac{c_{\omega s}}{\mathfrak{v}} = \frac{\varphi_{\omega s}}{\mathscr{V}} = \frac{l_s}{r_w} = L_s = \frac{\alpha}{2}, \quad \text{(A62a)}$$

or

$$\frac{c_{\omega ws}}{\mathfrak{v}} = \frac{\varphi_{\omega ws}}{\mathscr{V}} = \frac{l_{ws}}{r_w} = L_{ws} = \frac{\alpha_w}{2}. \quad \text{(A62b)}$$

It follows from a comparison of equations (A49) and (A61) that

$$\frac{Re_\Gamma}{Re_{\omega s}} = \frac{Re_{\Gamma w}}{Re_{\omega ws}} = 2.$$

Similarly, we may introduce the fictitious reduced peripheral velocity $c_{\omega 0}$ or $c_{\omega 0\mathrm{w}}$ related to the radius $y = r_\mathrm{w}$, namely

$$c_{\omega 0} = \omega r_\mathrm{w} \tag{A63a}$$

or

$$c_{\omega 0\mathrm{w}} = \omega_\mathrm{w} r_\mathrm{w} \tag{A63b}$$

and then write the proportion

$$\frac{\varepsilon}{v} \sim 2(y_\omega^\mathrm{X} - y_{\omega\mathrm{w}}^\mathrm{X}), \tag{A64}$$

where

$$y_\omega^\mathrm{X} = \frac{y c_{\omega 0}}{v} \tag{A65a}$$

or

$$y_{\omega\mathrm{w}}^\mathrm{X} = \frac{y c_{\omega 0\mathrm{w}}}{v}. \tag{A65b}$$

The same procedure as in the case of eddy viscosity may be applied in the mathematical expression of the connection between the isolated vortex diffusion and the eddy diffusivity of heat. Analogously, we may write

$$\frac{\varepsilon_q}{a} \sim \frac{r_\mathrm{w} y}{a}(\Omega_q - \Omega_{q\mathrm{w}}). \tag{A66}$$

For liquid metals

$$t_{q\Gamma} = t_\Gamma\, Pr, \tag{A67a}$$

$$t_{q\Gamma\mathrm{w}} = t_{\Gamma\mathrm{w}}, \tag{A67b}$$

$$Fo_\Gamma = Zh_\Gamma\left(= \frac{1}{4\pi}\right), \tag{A67c}$$

$$Fo_{\Gamma\mathrm{w}} = \frac{Zh_{\Gamma\mathrm{w}}}{Pr}, \tag{A67d}$$

where $Fo_\Gamma = at_{q\Gamma}/r_\mathrm{w}^2$ or $Fo_{\Gamma\mathrm{w}} = at_{q\Gamma\mathrm{w}}/r_\mathrm{w}^2$ are Fourier numbers.

For classical fluids

$$\frac{2A\, Pr^{1/2}}{\alpha} = \frac{Re_\Gamma\, Pr}{4Fo_{\Gamma\mathrm{K}}}, \tag{A68a}$$

$$\frac{2A_\mathrm{w}\, Pr^{1/2}}{\alpha_\mathrm{w}} = \frac{Re_{\Gamma\mathrm{w}}\, Pr}{4Fo_{\Gamma\mathrm{wK}}}, \tag{A68b}$$

$$Fo_{\Gamma\mathrm{K}} = \frac{(va)^{1/2}}{Pr}\frac{t_{q\Gamma}}{r_\mathrm{w}^2} = \frac{at_{q\Gamma}}{Pr^{1/2} r_\mathrm{w}^2}, \tag{A69a}$$

$$Fo_{\Gamma\mathrm{wK}} = \frac{(va)^{1/2}}{Pr}\frac{t_{q\Gamma\mathrm{w}}}{r_\mathrm{w}^2} = \frac{at_{q\Gamma\mathrm{w}}}{Pr^{1/2} r_\mathrm{w}^2}, \tag{A69b}$$

where $Fo_{\Gamma\mathrm{K}}$ and $Fo_{\Gamma\mathrm{wK}}$ are modified Fourier numbers.

Summary of the models

For both the eddy viscosity model and the eddy diffusivity of heat model there are three suitable physical interpretations which do not contradict each other. Also the dimensionless complex $2A/\alpha$ or $2A_\mathrm{w}/\alpha_\mathrm{w}$ allows different interpretations.

(1) Eddy viscosity may be interpreted as the result of a mixing process characterized by the constant transverse mixing velocity and by a certain characteristic length which may be split into two components. The local magnitude of both the components is dependent on the distance from the wall while the course of the mixing length in the core is practically identical for all regimes of developed flow. Much closer to reality is the idea of two mixing processes with the mixing velocities of the same magnitude but of the opposite direction ($\mathfrak{v}_\mathrm{w} = -\mathfrak{v}$) and with different local values of the mixing lengths ($l_\mathrm{w} < l$). The complex $2A/\alpha$ or $2A_\mathrm{w}/\alpha_\mathrm{w}$ expresses the dimensionless mixing velocity in the form identical with Reynolds number in considering the mixing velocity as characteristic

$$\frac{2A}{\alpha} = \frac{2\mathfrak{v} r_\mathrm{w}}{v} = Re_\varepsilon. \tag{A70}$$

(2) The mathematical description of the model is identical with the mathematical description of diffusion of two co-axial isolated vortices of opposite direction with axis on the wall, 'frozen' at certain times. The time of stopping the basic vortex expressed in the dimensionless form Zh_Γ is constant for the developed flow, while for the other vortex the time of stopping decreases with increasing Reynolds number. The complex $2A/\alpha$ may then be expressed by the ratio of the dimensionless circulation velocity Re_Γ or $Re_{\Gamma\mathrm{w}}$ and the dimensionless time Zh_Γ or $Zh_{\Gamma\mathrm{w}}$ until the fictitious stopping of the vortex

$$\frac{2A}{\alpha} = \frac{Re_\Gamma}{4Zh_\Gamma} = \frac{Re_{\Gamma\mathrm{w}}}{4Zh_{\Gamma\mathrm{w}}}. \tag{A71}$$

This ratio corresponds to the dimensionless time change of the circulation velocity. For the dimensionless expression of the time change of the velocity, namely by the mean value of retardation $c_{\Gamma 0}/2t_\Gamma = c_{\Gamma 0\mathrm{w}}/2t_{\Gamma\mathrm{w}}$ it is necessary to supply the factor r_w^3/v^2

$$\frac{2A}{\alpha} = \frac{c_{\Gamma 0}}{2t_\Gamma}\frac{r_\mathrm{w}^3}{v^2} = \frac{c_{\Gamma 0\mathrm{w}}}{2t_{\Gamma\mathrm{w}}}\frac{r_\mathrm{w}^3}{v^2}. \tag{A72}$$

(3) The statistical conception of the problem, on the basis of which the eddy viscosity model has been derived, appears to be the most suitable approximation. Eddy viscosity is viewed as the result of two stochastic processes, the basic one being limited by the dimensions of the channel, with its basic parameters subsequently constant ($\alpha = 1/\pi$) for all regimes of developed flow. The other process, which is of a lesser reach (its effect being dominant in the wall region) depends on the Reynolds number.

Analogously as in the case of the physical idea of the two diffusing vortices, the time scales of the stochastic processes may be obtained by replacing the dispersion expressed dimensionally, $\sigma_y^2 = \sigma_Y^2 r_\mathrm{w}^2$, by the product of time t_σ and kinematic viscosity v,

$$\sigma_y^2 = t_\sigma v \tag{A73}$$

or by replacing the dimensionless dispersion σ_Y^2 directly by Zhukowsky number Zh_σ

$$\sigma_Y^2 = Zh_\sigma = \frac{vt_\sigma}{r_\mathrm{w}^2} = = \frac{4-\pi}{4}\alpha, \tag{A74a}$$

$$\sigma_{Y\mathrm{w}}^2 = Zh_{\sigma\mathrm{w}} = \frac{vt_{\sigma\mathrm{w}}}{r_\mathrm{w}^2} = \frac{4-\pi}{4}\alpha_\mathrm{w}. \tag{A74b}$$

There is a proportionality between the time t_Γ or $t_{\Gamma\mathrm{w}}$ in expressing eddy viscosity by means of two 'frozen' vortices and the time scales t_σ or $t_{\sigma\mathrm{w}}$

$$\frac{t_\sigma}{t_\Gamma} = \frac{t_{\sigma\mathrm{w}}}{t_{\Gamma\mathrm{w}}} = \frac{Zh_\sigma}{Zh_\Gamma} = \frac{Zh_{\sigma\mathrm{w}}}{Zh_{\Gamma\mathrm{w}}} = \frac{4-\pi}{1} = 0.8584073461. \tag{A75}$$

For the relative eddy viscosity we may then write, analogically to equation (A44)

$$\begin{aligned}\frac{\varepsilon}{v} = \frac{Re_\sigma}{[4/(4-\pi)]Zh_\sigma}&[Y(\mathrm{e}^{-Y^2/[4/(4-\pi)]Zh_\sigma} - \mathrm{e}^{-Y^2/[4/(4-\pi)]Zh_{\sigma\mathrm{w}}}) \\ &+(2-Y)(\mathrm{e}^{-(2-Y)^2/[4/(4-\pi)]Zh_\sigma} - \mathrm{e}^{-(2-Y)^2/[4/(4-\pi)]Zh_{\sigma\mathrm{w}}}) \\ &-(2+Y)(\mathrm{e}^{-(2+Y)^2/[4/(4-\pi)]Zh_\sigma} - \mathrm{e}^{-(2+Y)^2/[4/(4-\pi)]Zh_{\sigma\mathrm{w}}}) \\ &-(4-Y)(\mathrm{e}^{-(4-Y)^2/[4/(4-\pi)]Zh_\sigma} - \mathrm{e}^{-(4-Y)^2/[4/(4-\pi)]Zh_{\sigma\mathrm{w}}})],\end{aligned} \tag{A76}$$

where

$$Re_\sigma = Re_\Gamma = 2A.$$

To the value $\alpha = 1/\pi$ there is the corresponding dispersion $\sigma_Y^2 = Zh_\sigma = (4-\pi)/4\pi = 0.06830988614 = \text{const.}$ and the standard deviation $\sigma_Y = (\sigma_Y^2)^{1/2} = 0.261361004 = \text{const.}$ The coefficient α_w, however, depends to a considerable extent on Reynolds number Re, or on the complex $(f/4)Re$. The values of the time scales $Zh_{\sigma w} = \sigma_{Yw}^2$ are given in Table 6 together with the mean values of the Rayleigh distribution η_w according to equation (3.11).

The formal agreement between the stochastic model of eddy viscosity and the velocity field of the 'frozen' isolated vortices allows the complex $2A/\alpha$ to be written as

$$\frac{2A}{\alpha} = \frac{Re_\sigma}{[4/(4-\pi)]Zh_\sigma} = \frac{Re_{\sigma w}}{[4/(4-\pi)]Zh_{\sigma w}}, \quad \text{(A77a)}$$

where

$$Re_\sigma = Re_\Gamma = 2A$$

and

$$Re_{\sigma w} = Re_{\Gamma w} = 2A_w$$

or

$$\frac{2A}{\alpha} = \frac{4-\pi}{2}\frac{c_{\sigma 0}}{t_\sigma}\frac{r_w^3}{v^2} = \frac{4-\pi}{2}\frac{c_{\sigma 0w}}{t_{\sigma w}}\frac{r_w^3}{v^2}, \quad \text{(A77b)}$$

where $c_{\sigma 0} = c_{\Gamma 0}$ and $c_{\sigma 0w} = c_{\Gamma 0w}$ is the fictitious circulation velocity and the ratios

$$\frac{4-\pi}{2}\frac{c_{\sigma 0}}{t_\sigma} = \frac{4-\pi}{2}\frac{c_{\sigma 0w}}{t_{\sigma w}} = \frac{c_{\Gamma 0w}}{2t_{\Gamma w}} = \frac{c_{\Gamma 0}}{2t_\Gamma}$$

are the mean value of retardation.

The equations (A72) and (A77), expressing the complex $2A/\alpha$, are formally identical with

Galileo number $Ga = g(r_w^3/v^2)$,
Archimedes number $Ar = g(\Delta\rho/\rho_0)(r_w^3/v^2)$ and
Grashof number $Gr = g(\Delta T/T_0)(r_w^3/v^2)$

in which the first term corresponds to the velocity variation in time, namely the acceleration or reduced acceleration, and the term r_w^3/v^2 is identical in all the criteria.

Conclusions analogical to those drawn from the eddy viscosity evaluation from the point of view of mathematical statistics, may also be reached in the case of the eddy diffusivity of heat. To obtain the time scales of the processes in turbulent heat transport, namely Fourier numbers Fo_σ and $Fo_{\sigma w}$, it is sufficient to replace the value of the coefficient 4 at Fo_Γ or $Fo_{\Gamma w}$ in equations (A67c) and (A68a,b) by the value $4/(4-\pi)$. The equation (A75) may also be applied to heat transport, resulting in an analogical proportion

$$\frac{Fo_\sigma}{Fo_\Gamma} = \frac{Fo_{\sigma w}}{Fo_{\Gamma w}} = \frac{Fo_{K\sigma}}{Fo_{K\Gamma}} = \frac{Fo_{K\sigma w}}{Fo_{K\Gamma w}} = \frac{4-\pi}{1} = 0.858407346. \quad \text{(A78)}$$

The eddy viscosity model enables the decisive basic hydrodynamic characteristics of the developed turbulent fluid flow to be determined without the knowledge of any further experimental data.

With the use of the values given in Table 2 an approximate proportion may be found between the expression $(f/2)^{1/2}Re$, which represents the Reynolds friction number Re^*, and the complex $2A/\alpha$ (i.e. the dimensionless transverse mixing velocity)

$$\left(\frac{f}{2}\right)^{1/2} Re = Re^* = 2k\frac{2A}{\alpha} \quad \text{(A79a)}$$

or, in dimensional form,

$$u^* = \left(\frac{\tau_w}{\rho}\right)^{1/2} = 2kv \Rightarrow f = 8k^2 \psi^{-2}. \quad \text{(A79b)}$$

The coefficient of proportionality k ranges in the limit of the values of $k = 2.8$ for low Reynolds numbers and $k = 2.55$ for high Reynolds numbers. The mean value of the coefficient is $k = 2.675$. With the use of the preceding proportion the friction factor may be determined from the values of the coefficients α and α_w or A and A_w and from the values of the complex $(f/4)Re$:

$$\left(\frac{f}{2}\right)^{1/2} = \frac{(f/4)Re}{k(2A/\alpha)}. \quad \text{(A80)}$$

The proportionality between the transverse mixing velocity v and the fictitious turbulent friction velocity $u^*_{\tau\,max}$ determined from the maximum turbulent shear stress $\tau_{t\,max}$ instead of the shear stress at the wall τ_w displays better agreement,

$$u^*_{\tau\,max} = 2k_t v, \quad \text{(A81a)}$$

$$u^*_{\tau\,max} = \left(\frac{\tau_{t\,max}}{\rho}\right)^{1/2} = \left(\frac{\tau_w}{\rho}\right)^{1/2}\left(\frac{\tau_t}{\tau_w}\right)^{1/2}_{max} = u^*\left(\frac{\tau_t}{\tau_w}\right)^{1/2}_{max}. \quad \text{(A81b)}$$

The proportion in a dimensionless form is then

$$\left(\frac{f}{2}\right)^{1/2}\left(\frac{\tau_t}{\tau_w}\right)^{1/2}_{max} Re = 2k_t\frac{2A}{\alpha} \Rightarrow f = \frac{8k_t^2\psi^{-2}}{(\tau_t/\tau_w)_{max}}. \quad \text{(A81c)}$$

For the coordinate $Y = Y_{\tau\,max}$, obtained from the equation (3.43), and using equation (3.42), we may determine $(\tau_t/\tau_w)_{max}$.

If we use the values from Table 2 the proportionality coefficient k_t in the observed region of parameters will range from $k_t = 2.67$ for low Reynolds numbers to $k_t = 2.63$ for high Reynolds numbers Re. If the deviation of 0.7% is taken into account the value of the proportionality coefficient $k_t = 2.65$ can be taken as constant. An expression for the friction factor, f, [equation (A82a) which is analogous to equation (A80)] may be found

$$\left(\frac{f}{2}\right)^{1/2} = \frac{(\tau_t/\tau_w)^{1/2}_{max}(f/4)Re}{k_t(2A/\alpha)} \quad \text{(A82a)}$$

Table 6. Time scales and mean values of the stochastic reach of the wall into the fluid flow

	α_w				
	10^{-6}	10^{-5}	10^{-4}	10^{-3}	10^{-2}
η_w	0.0008862269255	0.002802495609	0.008862269255	0.02802495609	0.08862269255
$\sigma_{Yw}^2 = Zh_{\sigma w}$	2.146018365 $\times 10^{-7}$	2.146018365 $\times 10^{-6}$	2.146018365 $\times 10^{-5}$	2.146018365 $\times 10^{-4}$	2.146018365 $\times 10^{-3}$
σ_{Yw}	4.632513750 $\times 10^{-4}$	1.464929474 $\times 10^{-3}$	4.632513750 $\times 10^{-3}$	1.464929474 $\times 10^{-2}$	4.632513750 $\times 10^{-2}$
α		$1/\pi =$	0.3183098861		
η			0.5		
$\sigma_Y^2 = Zh_\sigma$			0.06830988614		
σ_Y			0.261361004		

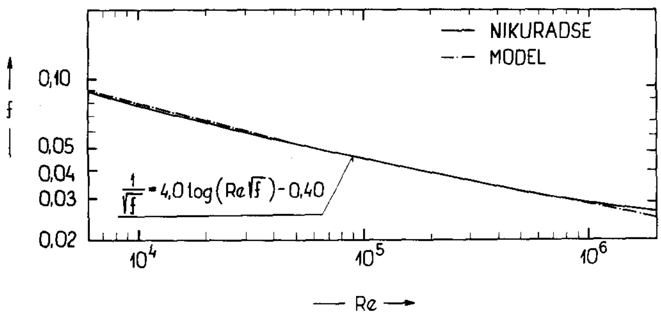

FIG. 28. Fanning friction factor $f = f(Re)$.

or

$$f = \frac{2(\tau_t/\tau_w)_{max}[(f/4)Re]^2}{k_t^2(2A/\alpha)^2} \tag{A82b}$$

with the corresponding value of the Reynolds number given by

$$Re = \frac{2k_t^2(2A/\alpha)^2}{(\tau_t/\tau_w)_{max}(f/4)Re}. \tag{A82c}$$

The values of the friction factor f, determined using the three coefficients A, α, α_w from equation (A82b), as a function of the respective Reynolds numbers Re are given in Fig. 28 together with the values by Nikuradse according to equation (3.27). The pairs of values of the friction factor f and Reynolds number Re are given in Table 5. The largest deviations from the Nikuradse relation, which do not exceed 1%, are found in the values of the friction factor f determined from equation (A82b) at very low and very high values of Reynolds number. Both the dependences are practically identical at $Re \sim 10^5$. A comparison with $f = f(Re)$, according to various authors, confirms the accuracy of the friction factor values in the range of Reynolds number $Re \in \langle 7 \times 10^3, 1.3 \times 10^6 \rangle$, as given above.

The model of eddy viscosity is in accordance with Prandtl's conception as well as the mathematical expression of Reynolds shear stress

$$\tau_t = \rho l^2 \left(\frac{du}{dy}\right)^2. \tag{A83}$$

It differs only in that in Prandtl's conception the transverse mixing velocity v is regarded as the product of the mixing length l and the velocity gradient du/dy [$v = l(du/dy)$]. In the model in question the velocity v is considered constant for the given flow regime.

MODÈLE DE LA VISCOSITÉ TURBULENTE ET DU COEFFICIENT D'ECHANGE THERMIQUE TURBULENT

Résumé—La connexion entre la viscosité turbulente et la distribution des fréquences de la portée de l'influence d'une paroi matérielle dans le courant d'un fluide est dérivée. L'expression de l'influence de la paroi est généralisée, valide aussi bien pour l'écoulement et la transmission de la chaleur dans des canaux lisses et rugueux, comme pour l'écoulement autour d'une surface. Le projet d'un modèle de la viscosité turbulente d'une circulation établie d'un fluide aux propriétés constantes dans un tube lisse est élaboré. Les coefficients apparaissant dans le modèle sont déterminées et à partir de ces coefficients les caractéristiques hydrodynamiques du courant sont déterminées. Le projet d'un modèle analogue de l'échange thermique turbulent est élaboré, la connexion entre les coefficients des deux modèles est déterminée et les caractéristiques thermocinétiques des fluides avec $Pr = 0.72$–10 avec $q_w =$ const. sont calculées. Le modèle du coefficient d'échange thermique turbulent est modifié pour les métaux liquides. L'amalgame des deux modèles permet de déterminer l'influence de l'énergie dissipée sur les proprietés thermocinétiques y compris le coefficient de la transmission de l'energie dissipée. Le sens physique des coefficients des modèles et leur connexion avec le parcours de mélange et les quantités qui caractérisent la diffusion du filet-tourbillon est discuté.

MODELL DER TURBULENTEN ZÄHIGKEIT UND TURBULENTEN TEMPERATURFÄHIGKEIT

Zusammenfasung—Abgeleitet ist der Zuzammenhang zwischen der turbulenten Zähigkeit und der Verteilung der Wahrscheinlichkeitdichte des Einflusses der Wand auf das strömende Medium. Der Ausdrück des Einflusses der Wand ist allgemein gültig für die Strömung und den Wärmeübergang in glatten und rauhen Kanälen und für die äussere Strömung der Oberflächen. Entworfen ist ein Modell der turbulenten Zähigkeit für entwickelte Strömung eines Mediums mit konstanten Eigenschaften in glatten Rohre. Die Koeffizienten des Modells sind festgestellt und bei ihren Anwendung hydrodynamische Charakteristiken der Strömung bestimmt. Ein analogisches Modell für die turbulente Temperaturleitfähigkeit ist entworfen, der Zusammenhang zwischen den Koeffizienten beider Modelle abgeleitet und für $q_w =$ konst. thermokinetische Charakteristiken für Flüssigkeiten mit $Pr = 0.72$–10 bestimmt. Das Modell ist gleichfalls für flüssige Metalle zubereitet. Die Verbindung beider Modelle erlaubt den Einfluss der Dissipationsenergie auf die thermokinetischen Charakteristiken und des Wärmeübergangkoeffizienten für die Dissipationsenergie zu bestimmen. Besprochen ist der physikalische Sinn der Koeffizienten des Modells sowie die Zusammenhänge mit der Mischungslänge und mit den Grössen, die die Wirbeldiffusion charakterisieren.

МОДЕЛЬ ВИХРЕВОЙ ВЯЗКОСТИ И ВИХРЕВОЙ ТЕМПЕРАТУРОПРОВОДНОСТИ

Аннотация — Предложена статистическая модель вихревой вязкости, рассматриваемая как распределение плотности вероятности влияния твердой стенки на обтекающий ее поток жидкости. Предложенные соотношения, учитывающие влияние стенки, по предположению справедливы для переноса импульса и тепла в гладких и шероховатых каналах и пристенных течений типа пограничного слоя. На основе этих соотношений получена модель вихревой вязкости для развитого течения жидкости с постоянными свойствами в гладких трубах. Найдены коэффициенты модели и представлены численные результаты расчета основных гидродинамических характеристик. Представлена аналогичная модель вихревой температуропроводности и показана ее связь с моделью вихревой вязкости. Рассчитаны тепловые характеристики турбулентного течения ($Pr = 0{,}72$–10) жидкости в трубе при однородном тепловом потоке на стенке. Модель вихревой температуропроводности распространена на случай жидких металлов. Приложение обеих моделей позволяет определить влияние диссипации энергии на гидродинамические и тепловые характеристики течения, а также коэффициент теплопереноса. Обсуждается физический смысл коэффициентов моделей и показана их связь с длиной пути смешивания.